Non-local Poisson structures and applications to the theory of integrable systems II

Alberto De Sole [*], Victor G. Kac [†]

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Abstract

We develop further the Lenard-Magri scheme of integrability for a pair of compatible non-local Poisson structures, which we discussed in Part I. We apply this scheme to several such pairs, proving thereby integrability of various evolution equations, as well as hyperbolic equations. Some of these equations may be new.

1 Introduction

In Part I of this series of papers we laid grounds of a rigorous theory of non-local Poisson structures. In particular we have developed the Lenard-Magri scheme of integrability for a pair of compatible non-local Poisson structures.

In Section 2 of the present Part II we made several improvements of this scheme, which facilitate its application in concrete examples. The concrete examples are studied in Sections 3, 4 and 5. In Section 3 we consider three compatible scalar non-local Poisson structures:

$$L_1 = \partial \,, \ L_2 = \partial^{-1} \,, \ L_3 = u' \partial^{-1} \circ u' \ (\text{ Sokolov [Sok84]})\,,$$

and take for a compatible pair (H, K) arbitrary linear combinations of these three structures: $H = \sum_i a_i L_i$, $K = \sum_i b_i L_i$. We study in detail for which values of the coefficients a_i and b_i the corresponding Lenard-Magri scheme is integrable. This means that there exists an infinite sequence of Hamiltonian

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[*]Dept. of Math., University of Rome "La Sapienza", Rome, Italy, and IHES, France, desole@mat.uniroma1.it

[†]Department of Math., M.I.T., Cambridge, MA 02139, USA. and IHES, France, kac@math.mit.edu

functionals $\int h_n$, $n \in \mathbb{Z}_+$, and Hamiltonian vector fields P_n, $n \in \mathbb{Z}_+$, such that we have

$$(1.1) \qquad \int 0 \xleftrightarrow{H} P_0 \xleftrightarrow{K} \int h_0 \xleftrightarrow{H} P_1 \xleftrightarrow{K} \int h_1 \xleftrightarrow{H} \dots ,$$

and the spans of the $\int h_n$'s and of the P_n's are infinite dimensional (the association relation $\int f \xleftrightarrow{H} P$ is recalled in Definition 2.5). This produces integrable equations $u_{t_n} = P_n$.

Furthermore we study when the infinite sequence (1.1) can be extended to the left. The most interesting cases are those when the sequence is "blocked" at some step P_{-n}, $n > 0$, to the left. This leads to some interesting integrable hyperbolic equations. As a result, we get two such equations

$$(1.2) \qquad u_{tx} = e^u - \alpha e^{-u} + \epsilon(e^u - \alpha e^{-u})_{xx} ,$$

where α and ϵ are 0 or 1, and

$$(1.3) \qquad u_{tx} = u + (u^3)_{xx} .$$

Of course, in the case when $\epsilon = 0$, equation (1.2) is the Liouville (respectively sinh-Gordon) equation if $\alpha = 0$ (resp. $\alpha = 1$). But for $\epsilon = 1$ equation (1.2) seems to be new. Equation (1.3), first discovered in [SW02], is called the short pulse equation. Its integrability was proved in [SS04]

In Section 4 we study, in a similar way, two liner combinations of the compatible non-local Poisson structures

$$L_1 = u' \partial^{-1} \circ u' , \quad L_2 = \partial^{-1} \circ u' \partial^{-1} \circ u' \partial^{-1} \ (\text{ Dorfman [Dor93]}) .$$

As a result we (re)prove integrability of the Krichever-Novikov equation

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} ,$$

and also, moving to the left, establish integrability of the following equation

$$\left(\frac{u_{tx}}{u_x}\right)_x = \frac{1}{2u_x} + \gamma u_x ,$$

where γ is a constant.

Finally, in Section 5 we study, in a similar way, three two-components non-local Poisson structures that are used in the study of the non-linear

Schroedinger equation (NLS), [TF86, Dor93, BDSK09, DSK12]. As a result, we establish integrability of the following generalization of NLS:

$$i\psi_t = \psi_{xx} + \alpha\psi|\psi|^2 + i\beta(\psi|\psi|^2)_x\,,$$

where α and β are constants (NLS is obtained for $\beta = 0$).

2 The Lenard-Magri scheme of integrability

2.1 Non-local Poisson structures on an algebra of differential functions

Let $R_\ell = \mathbb{F}[u_i^{(n)}\,|\,i \in I, n \in \mathbb{Z}_+]$ be the algebra of differential polynomials in the ℓ variables u_i, $i \in I = \{1,\ldots,\ell\}$, with the derivation ∂ defined by $\partial(u_i^{(n)}) = u_i^{(n+1)}$. The partial derivatives $\frac{\partial}{\partial u_i^{(n)}}$ are commuting derivations of R_ℓ, and they satisfy the following commutation relations with ∂:

(2.1) $$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial\right] = \frac{\partial}{\partial u_i^{(n-1)}} \quad \text{(the RHS is 0 if } n = 0)\;.$$

Recall from [BDSK09] that an *algebra of differential functions* is a differential algebra extension $\mathcal{V}$ of R_ℓ, endowed with commuting derivations

$$\frac{\partial}{\partial u_i^{(n)}} : \mathcal{V} \to \mathcal{V}\,, \quad i \in I,\, n \in \mathbb{Z}_+\,,$$

extending the usual partial derivatives on R_ℓ, such that only a finite number of $\frac{\partial f}{\partial u_i^{(n)}}$ are non-zero for each $f \in \mathcal{V}$, and such that the commutation rules (2.1) hold on $\mathcal{V}$.

A *field of differential functions* is a field which is an algebra of differential functions. In particular, it contains the field of rational functions in the variables $u_i^{(n)}$, which is itself a field of differential functions. Given a field of differential functions $\mathcal{V}$, it is easy to see that it can be extended by adding solutions of algebraic equations over $\mathcal{V}$, and also by functions of the form $F(\varphi_1,\ldots,\varphi_n)$, where F is an infinitely differentiable function in n variables and $\varphi_1,\ldots,\varphi_n$ lie in $\mathcal{V}$. (Note, though, that in general we cannot add to $\mathcal{V}$ solutions of linear differential equations. For example, a solution of the equation $f' = fu'$ is $f = e^u$, which can be added, while a non-zero solution

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of the equation $f' = fu$ can never be added, due to simple differential order considerations). We will use these facts in the examples further on.

Note that, if the algebra of differential functions is a domain, then its field of fractions is a field of differential functions, with the maps $\frac{\partial}{\partial u_i^{(n)}}$ defined in the obvious way.

We will assume throughout the paper that $\mathcal{V}$ is a field of differential functions in the variables $u_1, \ldots, u_\ell$. We denote by $\mathcal{C} = \{c \in \mathcal{V} \mid \partial c = 0\} \subset \mathcal{V}$ the subset of *constants*, and by

$$\mathcal{F} = \left\{ f \in \mathcal{V} \,\Big|\, \frac{\partial f}{\partial u_i^{(n)}} = 0 \ \text{ for all } i \in I, n \in \mathbb{Z}_+ \right\} \subset \mathcal{V}$$

the subset of *quasiconstants*. It is easy to see that $\mathcal{C} \subset \mathcal{F} \subset \mathcal{V}$ is a tower of field extensions.

Given $f \in \mathcal{V}$ which is not a quasiconstant, we say that it has *differential order* N, and we write $\mathrm{dord}(f) = N$, if $\frac{\partial f}{\partial u_i^{(N)}} \neq 0$ for some $i \in I$, and $\frac{\partial f}{\partial u_j^{(n)}} = 0$ for every $j \in I$ and $n > N$. By definition, the differential order of a non-zero quasiconstant is $-\infty$. We let $\mathcal{V}_N$ be the subfield of elements of differential order at most N. This gives an increasing sequence of subfields

$$(2.2) \qquad\qquad \mathcal{C} \subset \mathcal{F} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V},$$

such that $\partial \mathcal{V}_N \subset \mathcal{V}_{N+1}$. It is easy to show, using (2.1), that

$$(2.3) \qquad \partial \mathcal{V} \cap \mathcal{V}_N = \partial \mathcal{V}_{N-1} \ \text{ for } N \geq 1, \ \text{ and } \ \partial \mathcal{V} \cap \mathcal{V}_0 = \partial \mathcal{F}.$$

Given an arbitrary $k \times \ell$-matrix A with entries in $\mathcal{V}$, we define its differential order, denoted by $\mathrm{dord}(A)$, as the maximal differential order of all its entries.

Given a matrix differential operator $D = \sum_{i=0}^{n} A_i \partial^i \in \mathrm{Mat}_{k \times \ell} \mathcal{V}[\partial]$, we define its *differential order* as

$$(2.4) \qquad\qquad \mathrm{dord}(D) = \max\{\mathrm{dord}(A_1), \ldots, \mathrm{dord}(A_n)\},$$

which should not be confused with its *order*, defined as

$$(2.5) \qquad\qquad |D| = n \ \text{ if } A_n \neq 0.$$

(Note that the notion of order carries over to matrix pseudofferential operators, while the differential order is not defined in general).

Lemma 2.1. *Let $D \in \mathrm{Mat}_{k \times \ell} \mathcal{V}[\partial]$ be a matrix differential operator over $\mathcal{V}$ and let $F \in \mathcal{V}^\ell$. Then:*

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(a) $\mathrm{dord}(DF) \leq \max\{\mathrm{dord}(D), \mathrm{dord}(F) + |D|\}$.

(b) If D has non-degenerate leading coefficient and it satisfies $\mathrm{dord}(F) + |D| > \mathrm{dord}(D)$, then $\mathrm{dord}(DF) = \mathrm{dord}(F) + |D|$.

(c) If D has non-degenerate leading coefficient and it satisfies $\mathrm{dord}(DF) > \mathrm{dord}(D)$, then $\mathrm{dord}(DF) = \mathrm{dord}(F) + |D|$.

Proof. Let $D = \sum_{s=0}^{n} A_s \partial^s$. Clearly, for $f \in \mathcal{V}$ and $s \in \mathbb{Z}_+$, we have $\mathrm{dord}(f^{(i)}) = \mathrm{dord}(f) + i$. Hence, If $h > \max\{\mathrm{dord}(D), \mathrm{dord}(F) + |D|\}$, we have

$$\frac{\partial}{\partial u^{(h)}}(DF)_i = \sum_{j=1}^{\ell}\sum_{s=0}^{n} \frac{\partial}{\partial u^{(h)}}(A_s)_{ij} F_j^{(s)} = 0,$$

for every $i = 1, \ldots, k$, proving part (a). Furthermore, if $|D| = n$ and $\mathrm{dord}(F) + n > \mathrm{dord}|D|$, we can use (2.1) to get

$$\frac{\partial}{\partial u^{(\mathrm{dord}(F)+n)}}(DF)_i = \sum_{j=1}^{\ell}\sum_{s=0}^{n} \frac{\partial}{\partial u^{(\mathrm{dord}(F)+n)}}(A_s)_{ij} F_j^{(s)}$$

$$= \sum_{j=1}^{\ell}(A_n)_{ij} \frac{\partial F_j^{(n)}}{\partial u^{(\mathrm{dord}(F)+n)}} = \sum_{j=1}^{\ell}(A_n)_{ij} \frac{\partial F_j}{\partial u^{(\mathrm{dord}(F))}}.$$

Since, by assumption, the leading coefficient $A_n \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}$ of D is non-degenerate, the RHS above is non-zero for some i. Hence, $\mathrm{dord}(DF) = \mathrm{dord}(F) + n$, proving part (b). Part (c) follows from parts (a) and (b). $\square$

For $f \in \mathcal{V}$, as usual we denote by $\int f$ the image of f in the quotient space $\mathcal{V}/\partial\mathcal{V}$. Recall that, by (2.1), we have a well-defined variational derivative $\frac{\delta}{\delta u} : \mathcal{V}/\partial\mathcal{V} \to \mathcal{V}^\ell$, given by

$$\frac{\delta \int f}{\delta u_i} = \sum_{n \in \mathbb{Z}_+}(-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}, \; i \in I.$$

Remark 2.2. Recall that $\mathrm{Ker}\left(\frac{\delta}{\delta u}\right) \supset \mathcal{F} + \partial\mathcal{V}$, and, by [BDSK09, Prop.1.5], equality holds if $\mathcal{V}$ is normal.

Denote by $\mathcal{V}((\partial^{-1}))$ the skewfield of pseudodifferential operators over $\mathcal{V}$. For $H(\partial) = \sum_{n=-\infty}^{N} h_n \partial^n \in \mathcal{V}((\partial^{-1}))$, we denote by $H(\lambda)$ its symbol: $H(\lambda) = \sum_{n=-\infty}^{N} h_n \lambda^n \in \mathcal{V}((\lambda^{-1}))$ (where λ is a variable commuting with $\mathcal{V}$). A closed formula for the associative product in $\mathcal{V}((\partial^{-1}))$ in terms of the corresponding symbols is the following:

(2.6) $$(A \circ B)(\lambda) = A(\lambda + \partial)B(\lambda).$$

Here and further we always expand an expression as $(\lambda + \partial)^n$, $n \in \mathbb{Z}$, in non-negative powers of ∂, so that, for $A(\partial) = \sum_{m=-\infty}^{N} a_m \partial^m$ and $B(\partial) = \sum_{n=-\infty}^{N} b_n \partial^n$, the RHS of (2.6) means $\sum_{m,n=-\infty}^{N} \sum_{k=0}^{\infty} \binom{m}{k} a_m b_n^{(k)} \lambda^{m+n-k}$.

An invertible element of $\mathrm{Mat}_{\ell \times \ell} \mathcal{V}((\partial^{-1}))$ is called non-degenerate. In particular, a matrix differential operator $A \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}[\partial]$ is called non-degenerate if it is invertible in $\mathrm{Mat}_{\ell \times \ell} \mathcal{V}((\partial^{-1}))$.

Denote by $\mathcal{V}(\partial)$ the minimal skew-subfield of the skew-field of pseudod-ifferential operators $\mathcal{V}((\partial^{-1}))$ containing the subalgebra $\mathcal{V}[\partial]$ of differential operators over $\mathcal{V}$. Elements of $\mathcal{V}(\partial)$ are called *rational* pseudodifferential operators. Elements of $\mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$ are called *rational matrix pseudodifferential operators.*

We often manipulate with rational pseudodifferential operators without expanding them as Laurent series in ∂^{-1}. Then their symbols are computed using formula (2.6).

Theorem 2.3 ([CDSK12b]). *Let $H(\partial) \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$. Then*

(a) *H has a fractional decomposition $H = AB^{-1}$, with $A, B \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}[\partial]$, and B non-degenerate, which is minimal in the sense that $\mathrm{Ker}(A)$ and $\mathrm{Ker}(B)$ have zero intersection in any differential field extension of $\mathcal{V}$.*

(b) *Any other fractional decomposition $H = \widetilde{A}\widetilde{B}^{-1}$ of H can be obtained by multiplying A and B on the right by the same element of $\mathrm{Mat}_{\ell \times \ell} \mathcal{V}[\partial]$.*

Let $H \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$ be an $\ell \times \ell$ rational matrix pseudodifferential operator. It is called a *Poisson structure* on $\mathcal{V}$ if it is skewadjoint and it satisfies the Jaocobi identity. In order to explain what the latter condition means, we introduce the non-local λ-bracket $\{\cdot_\lambda \cdot\} : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}((\lambda^{-1}))$ corresponding to H, given by the following *Master Formula* (cf. [DSK06])

$$(2.7) \qquad \{f_\lambda g\} = \sum_{\substack{i,j \in I \\ m,n \in \mathbb{Z}_+}} \frac{\partial g}{\partial u_j^{(n)}} (\lambda + \partial)^n H_{ji}(\lambda + \partial)(-\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}} .$$

In particular $\{u_{i\lambda} u_j\}_H = H_{ji}(\lambda)$, the symbol of the operator H. This λ-bracket satisfies the following properties:

(i) sesquilinearty: $\{\partial f_\lambda g\} = -\lambda \{f_\lambda g\}$, $\{f_\lambda \partial g\} = (\lambda + \partial)\{f_\lambda g\}$;

(ii) left Leibniz rule: $\{f_\lambda gh\} = \{f_\lambda g\}h + \{f_\lambda h\}g$;

(iii) right Leibniz rule: $\{fg_\lambda h\} = \{f_{\lambda+\partial} h\}_{\to} g + \{g_{\lambda+\partial} h\}_{\to} f$, where the arrow means that ∂ should be moved to the right.

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In fact, the above properties, rather than the Master Formula (2.7), are used in most computations of the λ-brackets.

For a matrix pseudodifferential operator $H(\partial) \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}((\partial^{-1}))$ and elements $f, g, h \in \mathcal{V}$, we have $\{f_\lambda \{g_\mu h\}\} \in \mathcal{V}((\lambda^{-1}))((\mu^{-1}))$, $\{g_\mu \{f_\lambda h\}\} \in \mathcal{V}((\mu^{-1}))((\lambda^{-1}))$, and $\{\{f_\lambda g\}_{\lambda+\mu} h\} \in \mathcal{V}(((\lambda+\mu)^{-1}))((\lambda^{-1}))$. On the other hand, if $H(\partial) \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$ is a rational matrix pseudodifferential operator, all these three elements lie in the same subspace $\mathcal{V}_{\lambda,\mu} = \mathcal{V}[[\lambda^{-1}, \mu^{-1}, (\lambda+\mu)^{-1}]][\lambda, \mu]$, [DSK12, Cor.3.11]. Hence, it makes sense to write the following *Jacobi identity*:

$$\{f_\lambda \{g_\mu h\}\} - \{g_\mu \{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda+\mu} h\}\,.$$

Note that the Jacobi identity holds for every $f, g, h \in \mathcal{V}$ if and only if it holds for every triple of generators u_i, u_j, u_k, [DSK12].

Definition 2.4. A *non-local Poisson structure* on $\mathcal{V}$ is a rational matrix pseudodifferential operator $H(\partial) \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$, which is skewadjoint, and satisfies the Jacobi identity for any triple of generators:

$$\{u_{i\lambda} \{u_{j\mu} u_k\}\} - \{u_{j\mu} \{u_{i\lambda} u_k\}\} = \{\{u_{i\lambda} u_j\}_{\lambda+\mu} u_k\}\,, \quad i, j, k \in I\,.$$

Two non-local Poisson structures H and K on $\mathcal{V}$ are *compatible* if any their linear combination $\alpha H + \beta K$, for $\alpha, \beta \in \mathcal{C}$ is a non-local Poisson structure.

Recall that, if $H(\partial) \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$ is a non-local Poisson structure on $\mathcal{V}$, then the corresponding non-local λ-bracket, given by the Master Formula (2.7), defines a structure of a *non-local Poisson vertex algebra* on $\mathcal{V}$, [DSK12].

2.2 Hamiltonian equations and integrability

Let $H \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$ be a non-local Poisson structure on the field of differential functions $\mathcal{V}$.

Definition 2.5. We say that elements $\int f \in \mathcal{V}/\partial \mathcal{V}$ and $P \in \mathcal{V}^\ell$ are *H-associated*, and we use the notation $\int f \xleftrightarrow{H} P$ (or $P \xleftrightarrow{H} \int f$), if there exist a fractional decomposition $H = AB^{-1}$ and an element $F \in \mathcal{V}^\ell$ such that

$$\frac{\delta f}{\delta u} = B(\partial) F\,, \quad P = A(\partial) F\,.$$

In this case, we say that $\int f$ is a *Hamiltonian functional* for H, and P is a *Hamiltonian vector field* for H.

Remark 2.6. Since, by Theorem 2.3, any fractional decomposition for H is obtained from a minimal one by multiplying both A and B on the right by the same matrix differential operator, it is clear that in the notion of H-association it suffices to use a fixed minimal fractional decomposition $H = A_0 B_0^{-1}$. In particular, the space of all Hamiltonian functionals for H is

$$(2.8) \qquad \mathcal{F}(H) = \left(\frac{\delta}{\delta u}\right)^{-1}(\operatorname{Im} B_0) \subset \mathcal{V}/\partial\mathcal{V},$$

and the space of all Hamiltonian vector fields for H is

$$(2.9) \qquad \mathcal{H}(H) = A_0\left(B_0^{-1}\left(\operatorname{Im}\frac{\delta}{\delta u}\right)\right) \subset \mathcal{V}^\ell.$$

Lemma 2.7. *(a) If $\int f \xleftrightarrow{H} P$ and $\int g \xleftrightarrow{H} Q$, then $\int (af + bg) \xleftrightarrow{H} (aP + bQ)$ for every $a, b \in \mathcal{C}$. In particular, $\mathcal{F}(H)$ and $\mathcal{H}(H)$ are vector spaces over $\mathcal{C}$.*

(b) If $\int f \xleftrightarrow{H} P$, then $\left\{\int g \in \mathcal{F}(H) \,\middle|\, \int g \xleftrightarrow{H} P\right\} = \int f + \mathcal{F}_0(H)$, where

$$(2.10) \qquad \mathcal{F}_0(H) = \left\{\int g \in \mathcal{F}(H) \,\middle|\, \int g \xleftrightarrow{H} 0\right\}.$$

(c) If $\int f \xleftrightarrow{H} P$, then $\left\{Q \in \mathcal{H}(H) \,\middle|\, \int f \xleftrightarrow{H} Q\right\} = P + \mathcal{H}_0(H)$, where

$$(2.11) \qquad \mathcal{H}_0(H) = \left\{Q \in \mathcal{H}(H) \,\middle|\, 0 \xleftrightarrow{H} Q\right\}.$$

Proof. Obvious. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 2.8 ([DSK12, Sec.7.1]). *(a) The space $\mathcal{V}^\ell$ has a Lie algebra structure with Lie bracket*

$$[P, Q]_i = \sum_{j \in I, n \in \mathbb{Z}_+} \frac{\partial Q_i}{\partial u_j^{(n)}} \partial^n P_j - \frac{\partial P_i}{\partial u_j^{(n)}} \partial^n Q_j,$$

and $\mathcal{H}(H) \subset \mathcal{V}^\ell$ is its subalgebra.

(b) We have a representation ϕ of the Lie algebra $\mathcal{V}^\ell$ on the space $\mathcal{V}/\partial\mathcal{V}$ given by

$$\phi(P)(\textstyle\int h) = \int P \cdot \frac{\delta h}{\delta u},$$

and the subspace $\mathcal{F}(H) \subset \mathcal{V}/\partial\mathcal{V}$ is preserved by the action of the Lie subalgebra $\mathcal{H}(H) \subset \mathcal{V}^\ell$.

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(c) If $\int h \xleftrightarrow{H} P$ and $\int h \xleftrightarrow{H} Q$ for some $\int h \in \mathcal{F}(H)$, then the action of $P, Q \in \mathcal{H}(H)$ on $\mathcal{F}(H)$ is the same:

$$\int P \cdot \frac{\delta g}{\delta u} = \int Q \cdot \frac{\delta g}{\delta u} \quad \text{for all } \int g \in \mathcal{F}(H).$$

(d) The space of Hamiltonian functionals $\mathcal{F}(H)$ is a Lie algebra with Lie bracket

$$(2.12) \qquad \{\int f, \int g\}_H = \int P \cdot \frac{\delta g}{\delta u} \quad \left(= \int \frac{\delta g}{\delta u} \cdot A(\partial) B^{-1}(\partial) \frac{\delta f}{\delta u} \right),$$

where $P \in \mathcal{H}(H)$ is such that $\int f \xleftrightarrow{H} P$.

(e) The Lie algebra action of $\mathcal{H}(H)$ on $\mathcal{F}(H)$ is by derivations of the Lie bracket (2.12).

(f) The subspace

$$\mathcal{A}(H) = \left\{ (\int f, P) \in \mathcal{F}(H) \times \mathcal{H}(H) \,\middle|\, \int f \xleftrightarrow{H} P \right\}$$

is a subalgebra of the direct product of Lie algebras $\mathcal{F}(H) \times \mathcal{H}(H)$.

A *Hamiltonian equation* associated to the Poisson structure H and to the Hamiltonian functional $\int h \in \mathcal{F}(H)$, with an associated Hamiltonian vector field $P \in \mathcal{H}(H)$, is, by definition, the following evolution equation on the variables $u = (u_i)_{i \in I}$:

$$(2.13) \qquad\qquad \frac{du}{dt} = P.$$

By the chain rule, any element $f \in \mathcal{V}$ evolves according to the equation

$$\frac{df}{dt} = \sum_{i \in I} \sum_{n \in \mathbb{Z}_+} (\partial^n P_i) \frac{\partial f}{\partial u_i^{(n)}},$$

and, integrating by parts, a local functional $\int f \in \mathcal{V}/\partial \mathcal{V}$ evolves according to

$$\frac{d \int f}{dt} = \int P \cdot \frac{\delta f}{\delta u}.$$

An *integral of motion* for the Hamiltonian equation (2.13) is a Hamiltonian functional $\int f \in \mathcal{F}(H)$ such that

$$\frac{d \int f}{dt} = \{\int h, \int f\}_H = 0,$$

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i.e. $\int f$ lies in the centralizer of $\int h$ in the Lie algebra $\mathcal{F}(H)$. The Hamiltonian equation (2.13) is said to be *integrable* if there is an infinite sequence of pairs $(\int h_n, P_n) \in \mathcal{F}(H) \times \mathcal{H}(H)$, $n \geq 0$, such that $\int h_0 = \int h$, $P_0 = P$, we have $\int h_n \xleftrightarrow{H} P_n$ for every $n \in \mathbb{Z}_+$, the sequence $\{\int h_n\}_{n \in \mathbb{Z}_+}$ spans an infinite dimensional abelian subalgebra of the Lie algebra $\mathcal{F}(H)$, and the sequence $\{P_n\}_{n \in \mathbb{Z}_+}$ spans an infinite dimensional abelian subalgebra of the Lie algebra $\mathcal{H}(H)$. Equivalently, equation (2.13) is integrable if there exists an abelian subalgebra $\mathfrak{h}$ of the Lie algebra $\mathcal{A}(H)$ defined in Proposition 2.8(f), such that both its canonical projections $\pi_1(\mathfrak{h})$ and $\pi_2(\mathfrak{h})$ in $\mathcal{F}(H)$ and $\mathcal{H}(H)$ respectively are infinite dimensional, and $\int h \in \pi_1(\mathfrak{h})$. In this case, we have an *integrable hierarchy* of Hamiltonian equations

$$\frac{du}{dt_n} = P_n , \ n \in \mathbb{Z}_+ .$$

Sometimes a non-local Poisson structure H admits a nice decomposition of the form

$$(2.14) \qquad\qquad H = A_1 B_1^{-1} A_2 B_2^{-1} \dots A_n B_n^{-1} ,$$

where $A_1, \dots, A_n, B_1, \dots, B_n$ are $\ell \times \ell$ matrix differential operators and the B_i's are-non degenerate. In this case, to check that $\int f \in \mathcal{V}/\partial\mathcal{V}$ and $P \in \mathcal{V}^\ell$ are H-associated one can use the following result.

Proposition 2.9. *Let $H \in \mathrm{Mat}_\ell \, \mathcal{V}(\partial)$ be a non-local Poisson structure of the form* (2.14). *Then we have $\int f \xleftrightarrow{H} P$, provided that there exist $F_1, \dots, F_n \in \mathcal{V}^\ell$ such that*
(2.15)

$$P = A_1 F_1 , \quad B_1 F_1 = A_2 F_2 , \quad \dots \quad , \ B_{n-1} F_{n-1} = A_n F_n , \quad B_n F_n = \frac{\delta f}{\delta u} .$$

For the proof we use the following:

Lemma 2.10 ([CDSK12c])**.** *Let A and B be $\ell \times \ell$ matrix differential operators, with B non-degenerate, and let $R = A\widetilde{B} = B\widetilde{A}$ be their right least common multiple. If $F, G \in \mathcal{V}^\ell$ solves $BF = AG$, then there exists $\widetilde{F} \in \mathcal{V}^\ell$ such that $F = \widetilde{B}\widetilde{F}$, $G = \widetilde{A}\widetilde{F}$.*

Proof of Proposition 2.9. For $n = 1$ the above condition (2.15) reduces to the Definition 2.5 of H-association, so there is nothing to prove. We prove the statement by induction on $n \geq 2$. Let $R = A_n \widetilde{B}_{n-1} = B_{n-1} \widetilde{A}_n$ be the

least right common multiple of A_n and B_{n-1}. Then, $B_{n-1}^{-1}A_n = \widetilde{A}_n \widetilde{B}_{n-1}^{-1}$, and the decomposition (2.14) gives

$$H = A_1 B_1^{-1} \ldots A_{n-2} B_{n-2}^{-1}(A_{n-1}\widetilde{A}_n)(B_n\widetilde{B}_{n-1})^{-1}.$$

By assumption, we have $B_{n-1}F_{n-1} = A_n F_n$. Hence, by Lemma 2.10, there exists $\widetilde{F}_{n-1} \in \mathcal{V}^\ell$ such that $F_{n-1} = \widetilde{A}_n\widetilde{F}_{n-1}$ and $F_n = \widetilde{B}_{n-1}\widetilde{F}_{n-1}$. It follows that $A_{n-1}F_{n-1} = (A_{n-1}\widetilde{A}_n)\widetilde{F}_{n-1}$, and $(B_n\widetilde{B}_{n-1})\widetilde{F}_{n-1} = B_n F_n = \frac{\delta f}{\delta u}$. Therefore, the assumption (2.15) implies $P = A_1 F_1$, $B_1 F_1 = A_2 F_2$, ... , $B_{n-2}F_{n-2} = (A_{n-1}\widetilde{A}_n)\widetilde{F}_{n-1}$, $(B_n\widetilde{B}_{n-1})\widetilde{F}_{n-1} = \frac{\delta f}{\delta u}$. By the inductive assumption, we conclude that $\int f \xleftrightarrow{H} P$. $\square$

2.3 The Lenard-Magri scheme of integrability

The so-called *Lenard-Magri scheme of integrability* provides an algorithm to construct hierarchies of integrable bi-Hamiltonian equations. Sufficient conditions for the applicability of this scheme are given in the following theorem, which summarizes the results in [DSK12, Sec.7.2].

Theorem 2.11. *Let H and $K \in \mathrm{Mat}_{\ell\times\ell}\mathcal{V}(\partial)$ be compatible non-local Poisson structures on a field of differential functions $\mathcal{V}$ (in ℓ variables $u_1, \ldots, u_\ell$), and assume that K is non-degenerate. Let, for some $N \in \mathbb{Z}_+$, $\{\int h_n\}_{n=0}^N \subset \mathcal{F}(H)$ and $\{P_n\}_{n=0}^N \subset \mathcal{H}(H)$ be such that*

$$(2.16) \quad \int 0 \xleftrightarrow{H} P_0 \xleftrightarrow{K} \int h_0 \xleftrightarrow{H} P_1 \xleftrightarrow{K} \int h_1 \xleftrightarrow{H} \ldots \xleftrightarrow{H} P_N \xleftrightarrow{K} \int h_N,$$

and such that the following orthogonality conditions hold for some fractional decompositions $H = AB^{-1}$ and $K = CD^{-1}$:

$$(2.17) \quad \Big(\mathrm{Span}_{\mathcal{C}}\Big\{\frac{\delta h_n}{\delta u}\Big\}_{n=0}^N\Big)^\perp \subset \mathrm{Im}(C), \quad \Big(\mathrm{Span}_{\mathcal{C}}\{P_n\}_{n=0}^N\Big)^\perp \subset \mathrm{Im}(B),$$

where the orthogonal complement is with respect to the pairing $\mathcal{V}^\ell \times \mathcal{V}^\ell \to \mathcal{V}/\partial\mathcal{V}$, $(\xi, P) \mapsto \int \xi \cdot P$. Then the given finite sequences can be extended to infinite sequences $\{\int h_n\}_{n=0}^\infty$, $\{P_n\}_{n=0}^\infty$, such that

$$(2.18) \qquad\qquad \int h_{n-1} \xleftrightarrow{H} P_n \xleftrightarrow{K} \int h_n \quad \text{for all } n \geq 1,$$

in some field of differential functions extension $\widetilde{\mathcal{V}}$ of $\mathcal{V}$. (In fact, the P_n's and $\frac{\delta h_n}{\delta u}$'s can be taken in $\mathcal{V}^\ell$.)

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The elements $\int h_n$, $n \in \mathbb{Z}_+$, are in involution with respect to both Lie brackets (2.12) for H and K (in the Lie algebras $\widetilde{\mathcal{F}}(H)$ and $\widetilde{\mathcal{F}}(K)$, subspaces of $\widetilde{\mathcal{V}}/\partial\widetilde{\mathcal{V}}$):

$$(2.19) \qquad \{\textstyle\int h_m, \int h_n\}_H = \{\textstyle\int h_m, \int h_n\}_K = 0 \ , \quad \text{for all } m, n \in \mathbb{Z}_+ \ ,$$

and,

$$(2.20) \qquad [P_m, P_n] \in \operatorname{Ker}(B^*) \cap \operatorname{Ker}(D^*) \ \ \text{for all } m, n \in \mathbb{Z}_+ \ .$$

In particular, if $\operatorname{Ker}(B^*) \cap \operatorname{Ker}(D^*) = 0$, then the evolution equations

$$(2.21) \qquad \frac{du}{dt_n} = P_n \ , \quad n \in \mathbb{Z}_+ \ ,$$

form a compatible hierarchy, and the $\int h_n$'s are its integrals of motion; therefore, if the subspaces $\operatorname{Span}_{\widetilde{\mathcal{C}}}\{\int h_n\}_{n\in\mathbb{Z}_+} \subset \widetilde{\mathcal{V}}/\partial\widetilde{\mathcal{V}}$ and $\operatorname{Span}_{\mathcal{C}}\{P_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}^\ell$ are infinite dimensional, then (2.21) is an integrable hierarchy of Hamiltonian equations.

Remark 2.12. In [DSK12, Sec.7.2] the above result is stated under the assumption that the algebra of differential functions $\mathcal{V}$ is normal. The present formulation is basically equivalent due to the fact that any algebra of differential functions can be extended to a normal one. Moreover, in the same paper we assume that the fractional decompositions $H = AB^{-1}$ and $K = CD^{-1}$ are minimal. However, it is clear that, if Theorem 2.11 holds for minimal fractional decompositions $H = A_0 B_0^{-1}$, $K = C_0 D_0^{-1}$, then it automatically holds for arbitrary fractional decompositions $H = AB^{-1}$, $K = CD^{-1}$. Indeed, thanks to Theorem 2.3, $\operatorname{Im} C \subset \operatorname{Im} C_0$, $\operatorname{Im} B \subset \operatorname{Im} B_0$, $\operatorname{Ker} B_0^* \subset \operatorname{Ker} B^*$, $\operatorname{Ker} D_0^* \subset \operatorname{Ker} D^*$.

Remark 2.13. The condition that $\operatorname{Ker}(B^*) \cap \operatorname{Ker}(D^*) = 0$ is actually not needed for the proof of integrability of each given equation $\frac{du}{dt} = P_n$. Indeed, since $\operatorname{Ker}(B^*) \cap \operatorname{Ker}(D^*)$ is finite dimensional over $\mathcal{C}$, then the space $\operatorname{Span}_{\mathcal{C}}\{P_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}^\ell$ contains an infinite dimensional abelian subalgebra containing any given element of it. This is true due to the following result.

Lemma 2.14. *Let U be an infinite dimensional subspace of a Lie algebra such that $[U, U]$ is finite dimensional. Then any element of U is contained in an infinite dimensional abelian subalgebra of U.*

Proof. Let a_1 be a non-zero element of U. The centralizer C_1 of a_1 in U is the kernel of the map $\operatorname{ad} a : U \to [U, U]$, hence, it has finite codimension in

U. Next, let a_2 be an element of C_1 linearly independent of a_1, and let C_2 be its centralizer in C_1. By the same argument, C_2 has finite codimension in C_1. In this fashion we construct an infinite sequence of linearly independent commuting elements of U. □

According to Theorem 2.11, when we apply the Lenard-Magri scheme of integrability we get infinite sequences $\{\int h_n\}_{n=0}^{\infty} \subset \mathcal{V}/\partial\mathcal{V}$, $\{P_n\}_{n=0}^{\infty} \subset \mathcal{V}^\ell$, satisfying the relations (2.18). At this point, in order to prove integrability, we still need to prove that the spaces $\mathrm{Span}_{\mathcal{C}}\{\int h_n\}_{n=0}^{\infty}$ and $\mathrm{Span}_{\mathcal{C}}\{P_n\}_{n=0}^{\infty}$ are infinite dimensional. (In fact, it is easy to see that it suffices to check the latter condition). For this one usually uses differential order considerations, based on the following result.

Proposition 2.15. *Suppose that the rational $\ell \times \ell$-matrix pseudodifferential operators $H = AB^{-1}$, $K = CD^{-1} \in \mathrm{Mat}_{\ell \times \ell}\, \mathcal{V}(\partial)$, are in their minimal fractional decompositions. Assume also that the matrix differential operators A, B, C, D have non-degenerate leading coefficients. Let $P, Q \in \mathcal{V}^\ell$ and $\int h \in \mathcal{V}/\partial\mathcal{V}$ satisfy the following relations*

$$(2.22) \qquad\qquad P \xleftrightarrow{K} \int h \xleftrightarrow{H} Q\,,$$

and assume that
(2.23)
$$\mathrm{dord}(P) > \max\{\mathrm{dord}(A) - |H| + |K|, \mathrm{dord}(B) + |K|, \mathrm{dord}(C), \mathrm{dord}(D) + |K|\}.\,\blacksquare$$

(Here we use the notation introduced in (2.4) and (2.5)). Then

$$\mathrm{dord}\left(\frac{\delta h}{\delta u}\right) = \mathrm{dord}(P) - |K| \quad and \quad \mathrm{dord}(Q) = \mathrm{dord}(P) + |H| - |K|\,.$$

Proof. By definition, the relations (2.22) amount to the existence of elements $F, G \in \mathcal{V}^\ell$ such that

$$CG = P\,, \quad DG = \frac{\delta h}{\delta u}\,, \quad BF = \frac{\delta h}{\delta u}\,, \quad AF = Q\,.$$

Since C has non-degenerate leading coefficient and $\mathrm{dord}(P) = \mathrm{dord}(CG) > \mathrm{dord}(C)$, we get by Lemma 2.1(c) that

$$\mathrm{dord}(G) = \mathrm{dord}(CG) - |C| = \mathrm{dord}(P) - |C|\,.$$

Next, since by assumption D has non-degenerate leading coefficient and $\mathrm{dord}(G) + |D| = \mathrm{dord}(P) - |C| + |D| = \mathrm{dord}(P) - |K| > \mathrm{dord}(D)$, we get

by Lemma 2.1(b) that

$$\operatorname{dord}(\frac{\delta h}{\delta u}) = \operatorname{dord}(DG) = \operatorname{dord}(G) + |D| = \operatorname{dord}(P) - |C| + |D|$$
$$= \operatorname{dord}(P) - |K| \,.$$

Similarly, since, by assumption, B has non-degenerate leading coefficient and $\operatorname{dord}(BF) = \operatorname{dord}(\frac{\delta h}{\delta u}) = \operatorname{dord}(P) - |K| > \operatorname{dord}(B)$, we get by Lemma 2.1(c) that

$$\operatorname{dord}(F) = \operatorname{dord}(BF) - |B| = \operatorname{dord}(\frac{\delta h}{\delta u}) - |B| = \operatorname{dord}(P) - |K| - |B| \,.$$

Finally, since, by assumption, A has non-degenerate leading coefficient and $\operatorname{dord}(F) + |A| = \operatorname{dord}(P) - |K| = |B| + |A| = \operatorname{dord}(P) - |K| + |H| > \operatorname{dord}(A)$, we get by Lemma 2.1(b) that

$$\operatorname{dord}(Q) = \operatorname{dord}(AF) = \operatorname{dord}(F) + |A| = \operatorname{dord}(P) - |K| - |B| + |A|$$
$$= \operatorname{dord}(P) - |K| + |H| \,.$$

$\square$

Recall from [DSK12, Theorem 5.1] that, given two compatible non-local Poisson structures H and K, with $\det(K) \neq 0$, we get an infinite family of compatible non-local Poisson structures given by $H^{[0]} = K$, and $H^{[s]} = (H \circ K^{-1})^{s-1} \circ H$, for $s \geq 1$.

Proposition 2.16. *Suppose that the sequences* $\{\int h_n\}_{n \in \mathbb{Z}_+} \subset \mathcal{F}(H) \cap \mathcal{F}(K)$, $\{P_n\}_{n \in \mathbb{Z}_+} \subset \mathcal{H}(H) \cap \mathcal{H}(K)$ *satisfy the Lenard-Magri recursive relations* $\int h_{n-1} \overset{H}{\longleftrightarrow} P_n \overset{K}{\longleftrightarrow} \int h_n$ *for every* $n \in \mathbb{Z}_+$ *(we let* $\int h_{-1} = \int 0$*). Then, we have* $\int h_n \overset{H^{[s]}}{\longleftrightarrow} P_{n+s}$ *for every* $n, s \in \mathbb{Z}_+$*. Consequently, all the evolution equations* $\frac{du}{dt_n} = P_n$ *are Hamiltonian with respect to all the non-local Poisson structures* $H^{[s]}$, $s \in \mathbb{Z}_+$.

Proof. By assumption we have $\int h_n \overset{H}{\longleftrightarrow} P_{n+1} \overset{K}{\longleftrightarrow} \ldots \overset{K}{\longleftrightarrow} \int h_{n+s-1} \overset{H}{\longleftrightarrow} P_{n+s}$, which means, by definition, that there exist $F_n, F_{n+1}, \ldots, F_{n+2s-2} \in \mathcal{V}^\ell$ such that

$$\frac{\delta h_n}{\delta u} = BF_n \,, \quad AF_{n+2i-2} = P_{n+i} = CF_{n+2i-1} \quad \text{for } 1 \leq i \leq s-1 \,,$$
$$DF_{n+2i-1} = \frac{\delta h_{n+i}}{\delta u} = BF_{n+2i} \quad \text{for } 1 \leq i \leq s-1 \,, \quad AF_{n+2s-2} = P_{n+s} \,.$$

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On the other hand, the non-local Poisson structure $H^{[s]}$ admits the decomposition

$$H^{[s]} = AB^{-1}DC^{-1} \ldots AB^{-1}DC^{-1}AB^{-1} \, ,$$

i.e. it is of the form (2.14) with $n = 2s + 1$, and with $A_i = A$ and $B_i = B$ for odd i, and $A_i = D$ and $B_i = C$ for even i. The claim follows from Proposition 2.9. $\qquad\square$

2.4 Lenard-Magri schemes of S-type and C-type

Consider a Lenard-Magri scheme as in (2.16). We say that it is *integrable* if it can be extended indefinitely, and the linear spans of $\{\int h_n\}_{n\in\mathbb{Z}_+}$ and $\{P_n\}_{n\in\mathbb{Z}_+}$ are infinite dimensional; in this case, the corresponding hierarchy of evolution Hamiltonian equations $\frac{du}{dt_n} = P_n$, $n \in \mathbb{Z}_+$ is integrable. We say that the Lenard-Magri scheme (2.16) is *finite* if it can be extended indefinitely, but in in any such infinite extension the linear span of $\{\int h_n\}_{n\in\mathbb{Z}_+}$ or of $\{P_n\}_{n\in\mathbb{Z}_+}$ is finite dimensional. Moreover, we say that the Lenard-Magri scheme (2.16) is *blocked* if it cannot be extended indefinitely, namely, for some n, there is no $\int h_n$ such that $P_n \xleftrightarrow{K} \int h_n$, or there is no P_{n+1} such that $\int h_n \xleftrightarrow{H} P_{n+1}$.

For an integrable Lenard-Magri scheme (2.16), we say that it is of *S-type* the differential orders of the elements P_n can grow to infinity, and it is of *C-type* if the differential orders of the P_n's are necessarily bounded. It is easy to see that for an integrable Lenard-Magri scheme of S-type the order of the pseudodifferential H should be greater than the order of the pseudodifferential K. Indeed, since we have $P_n \xleftrightarrow{K} \int h_n \xleftrightarrow{H} P_{n+1}$, if P_n has differential order large enough, then, by Proposition 2.15, $\mathrm{dord}(P_{n+1}) = \mathrm{dord}(P_n) + \mathrm{ord}(H) - \mathrm{ord}(K)$.

Remark 2.17. This terminology in inspired by the terminology of Calogero, who calls an integrable hierarchy of "S-type" if the differential orders of the canonical conserved densities are unbounded, and of "C-type" otherwise (see [MSS90, MS12]). Note that, though, these two terminologies are closed but do not coincide. For example, the linear equation $\frac{du}{dt} = u'''$ is C-integrable in Calogero's terminology, but the corresponding Lenard-Magri scheme, considered for example in [BDSK09], is integrable of S-type.

3 Liouville type integrable systems

In this section $\mathcal{V}$ is a field of differential functions in u, and we assume that $\mathcal{V}$ contains all the functions that we encounter in our computations.

Recall from [DSK12, Example 4.6] that we have the following triple of compatible non-local Poisson structures:

$$L_1 = \partial\,, \ L_2 = \partial^{-1}\,, \ L_3 = u'\partial^{-1} \circ u'\,.$$

Given two non-local Poisson structures H and K of the form

(3.1) $$H = a_1 L_1 + a_2 L_2 + a_3 L_3\,, \quad K = b_1 L_1 + b_2 L_2 + b_3 L_3\,,$$

with $a_i, b_i \in \mathcal{C}$, $i = 1, 2, 3$, we want to discuss the integrability of the corresponding Lenard-Magri scheme.

3.1 Preliminary computations

First, we find a minimal fractional decomposition for the operators H and K.

Lemma 3.1. *For $X = x_1 L_1 + x_2 L_2 + x_3 L_3$, with $x_1, x_2, x_3 \in \mathcal{C}$, we have*

(3.2) $$X = \left[x_1 \partial^2 \circ \frac{1}{u''}\partial + \frac{x_2 + x_3(u')^2}{u''}\partial - x_3 u' \right]\left[\partial \circ \frac{1}{u''}\partial \right]^{-1}\,.$$

The above fractional decomposition is minimal only for $x_2 x_3 \neq 0$. For $x_2 \neq 0$, $x_3 = 0$, the minimal fractional decomposition is

(3.3) $$X = (x_1 \partial^2 + x_2)\partial^{-1}\,,$$

for $x_2 = 0$, $x_3 \neq 0$ it is

(3.4) $$X = \left(x_1 \partial \circ \frac{1}{u'}\partial + x_3 u' \right)\left(\frac{1}{u'}\partial \right)^{-1}\,,$$

and for $x_2 = x_3 = 0$ it is $X = x_1 \partial$.

Proof. Straightforward. □

Later we will need the following simple facts concerning the numerators of the fractional decompositions for X.

Lemma 3.2. *(a) For $x_1, x_2, x_3 \in \mathcal{C}$, $x_1 \neq 0$, consider the equation*

(3.5) $$\left(x_1 \partial^2 \circ \frac{1}{u''}\partial + \frac{x_2 + x_3(u')^2}{u''}\partial - x_3 u' \right)F = f\,,$$

in the variables $F \in \mathcal{V}$ and $f \in \mathcal{F}$. If $x_3 \neq 0$, then all the solutions of equation (3.5) are

$$F = \alpha u'\,, \quad f = x_2 \alpha \quad \text{for some } \alpha \in \mathcal{C}\,,$$

while if $x_3 = 0$, then all the solutions of equation (3.5) are

$$F = \alpha u' + \beta(xu' - u) + \gamma\,, \quad f = x_2(\alpha + \beta x) \quad \text{for some } \alpha, \beta, \gamma \in \mathcal{C}\,.$$

(b) *For $x_1, x_2 \in \mathcal{C}$, $x_1 \neq 0$, an element $F \in \mathcal{V}$ satisfies*

(3.6) $$(x_1 \partial^2 + x_2) F \in \mathcal{F}$$

if and only if $F \in \mathcal{F}$.

(c) *For $x_1, x_3 \in \mathcal{C}$, $x_1 \neq 0$, an element $F \in \mathcal{V}$ satisfies*

(3.7) $$\left(x_1 \partial \circ \frac{1}{u'} \partial + x_3 u' \right) F \in \mathcal{V}_1$$

if and only if $F \in \mathcal{V}_0$ and $F' = \frac{\partial F}{\partial u} u'$.

Proof. If $n \geq 2$ and $F \in \mathcal{V}$ solves equation (3.5) and has differential order less than or equal to n, then, using (2.1), we have

$$0 = \frac{\partial}{\partial u^{(n+3)}} LHS(3.5) = x_1 \frac{1}{u''} \frac{\partial F}{\partial u^{(n)}},$$

implying that $\frac{\partial F}{\partial u^{(n)}} = 0$. Hence F must have differential order at most 1. Then we have

$$0 = \frac{\partial}{\partial u^{(4)}} LHS(3.5) = x_1 \left(\frac{1}{u''} \frac{\partial F}{\partial u'} - \frac{1}{(u'')^2} F' \right),$$

so that $F' = \frac{\partial F}{\partial u'} u''$. But then equation (3.5) becomes

(3.8) $$x_1 \left(\frac{\partial F}{\partial u'} \right)'' + (x_2 + x_3 (u')^2) \frac{\partial F}{\partial u'} - x_3 u' F = f.$$

If $\frac{\partial F}{\partial u'}$ has differential order $n \geq 0$, then applying $\frac{\partial}{\partial u^{(n+2)}}$ to both sides of equation (3.8) we get $\frac{\partial^2 F}{\partial u^{(n)} \partial u'} = 0$. Hence, $\frac{\partial F}{\partial u'} = \varphi$ must be a quasiconstant. In other words, $F = \varphi u' + f_0$, where $f_0 \in \mathcal{V}_0$ has differential order less than or equal to 0. But then the condition $F' = \frac{\partial F}{\partial u'} u''$ becomes $\varphi' u' + f_0' = 0$. This implies, using (2.3), that $(f_0 + \varphi' u)' = \varphi'' u \in \partial \mathcal{V} \cap \mathcal{V}_0 = \partial \mathcal{F}$. So, necessarily, $\varphi'' = 0$. Hence, $\varphi = \alpha + \beta x$ and $f_0 = -\beta u + \gamma$, for some constants $\alpha, \beta, \gamma \in \mathcal{C}$. Putting these results together, we have

$$F = (\alpha + \beta x) u' - \beta u + \gamma,$$

and plugging back into equation (3.8) we get

$$x_2 (\alpha + \beta x) + x_3 \beta u u' - x_3 \gamma u' = f.$$

Since, by assumption, $f \in \mathcal{F}$, we obtain $\beta = \gamma = 0$ if $x_3 \neq 0$, completing the proof of part (a).

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For part (b) we just observe that, if $F \in \mathcal{V}_n$ for some $n \geq 0$ satisfies condition (3.6), then

$$0 = \frac{\partial}{\partial u^{(n+2)}} (x_1 F'' + x_2 F) = x_1 \frac{\partial F}{\partial u^{(n)}} \,.$$

Hence, F must be a quasiconstant.

Similarly, if $F \in \mathcal{V}_n$ for some $n \geq 1$ satisfies condition (3.7), then

$$0 = \frac{\partial}{\partial u^{(n+2)}} \left(x_1 \partial \frac{F'}{u'} + x_3 u' F \right) = \frac{x_1}{u'} \frac{\partial F}{\partial u^{(n)}} \,.$$

Hence, F must lie in $\mathcal{V}_0$. Furthermore,

$$0 = \frac{\partial}{\partial u''} \left(x_1 \partial \frac{F'}{u'} + x_3 u' F \right) = x_1 \left(-\frac{F'}{(u')^2} + \frac{1}{u'} \frac{\partial F}{\partial u} \right) .$$

Hence, F must be such that $F' = \frac{\partial F}{\partial u} u'$. $\qquad\square$

Next, we compute the spaces $\mathcal{F}_0(X)$ and $\mathcal{H}_0(X)$ defined in (2.10) and (2.11). Here and further we use the following notation: given two constants $x_i, x_j \in \mathcal{C}$, we let

$$(3.9) \qquad\qquad x_{ij} = \sqrt{-\frac{x_j}{x_i}} \in \mathcal{C} \,.$$

(We assume that the field $\mathcal{C}$ contains all such elements).

Lemma 3.3. *For $X = x_1 L_1 + x_2 L_2 + x_3 L_3$, we have:*

(a) $\mathcal{F}_0(X) = \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$ *if* $x_1 x_2 x_3 \neq 0$;

   $\mathcal{F}_0(X) = \mathcal{C} \int e^{x_{12}x} u + \mathcal{C} \int e^{-x_{12}x} u + \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$ *if* $x_1 x_2 \neq 0, x_3 = 0$;

   $\mathcal{F}_0(X) = \mathcal{C} \int e^{x_{13}u} + \mathcal{C} \int e^{-x_{13}u} + \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$ *if* $x_1 x_3 \neq 0, x_2 = 0$;

   $\mathcal{F}_0(X) = \mathcal{C} \int u + \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$ *if* $x_1 \neq 0$ *and* $x_2 = x_3 = 0$;

   $\mathcal{F}_0(X) = \mathcal{C} \int \sqrt{x_2 + x_3 (u')^2} + \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$, *if* $x_1 = 0, x_2 x_3 \neq 0$;

   $\mathcal{F}_0(X) = \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$ *if* $x_1 = 0$ *and* $x_2 = 0$ *or* $x_3 = 0$.

(b) $\mathcal{H}_0(X) = \mathcal{C} \oplus \mathcal{C} u'$ *if* $x_2 x_3 \neq 0$;

   $\mathcal{H}_0(X) = \mathcal{C}$ *if* $x_2 \neq 0, x_3 = 0$;

   $\mathcal{H}_0(X) = \mathcal{C} u'$ *if* $x_2 = 0, x_3 \neq 0$;

   $\mathcal{H}_0(X) = 0$ *if* $x_2 = x_3 = 0$.

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Proof. First, let us find all elements $P \in \mathcal{H}_0(X)$. By Remark 2.6, if $X = YZ^{-1}$ is a minimal fractional decomposition, we need to solve the following equations in $F, P \in \mathcal{V}$:

(3.10) $$ZF = 0 , \quad P = YF .$$

By Lemma 3.1, if $x_2 = x_3 = 0$, then $Y = x_1 \partial$ and $Z = 1$, so the only solution of (3.10) is given by $F = 0$, $P = 0$. If $x_2 \neq 0$, $x_3 = 0$, then $Y = x_1 \partial^2 + x_2$ and $Z = \partial$, so we get $F \in \mathcal{C}$ and $P \in \mathcal{C}$. Similarly, if $x_2 = 0$, $x_3 \neq 0$, then $Y = x_1 \partial \circ \frac{1}{u'} \partial + x_3 u'$ and $Z = \frac{1}{u'} \partial$, so we get $F \in \mathcal{C}$ and $P \in \mathcal{C}u'$. Finally, if $x_2 \neq 0$, $x_3 \neq 0$, then $Y = x_1 \partial^2 \circ \frac{1}{u''} \partial + \frac{x_2 + x_3(u')^2}{u''} \partial - x_3 u'$ and $Z = \partial \circ \frac{1}{u''} \partial$. Hence, the solutions of (3.10) are $F = \alpha + \beta u' \in \mathcal{C} \oplus \mathcal{C}u'$, and $P = YF = x_2 \beta - x_3 \alpha u' \in \mathcal{C}u'$. This proves part (b).

Next, we find all elements $\int f \in \mathcal{F}_0(X)$, namely all solutions of the following equations in $F \in \mathcal{V}$ and $\int f \in \mathcal{V}/\partial\mathcal{V}$:

(3.11) $$YF = 0 , \quad \frac{\delta f}{\delta u} = ZF .$$

If $x_1 = 0, x_2 \neq 0, x_3 = 0$, we have $Y = x_2$ is invertible, and similarly, if $x_1 = 0, x_2 = 0, x_3 \neq 0$, we have $Y = x_3 u'$ is invertible too. In both these cases we thus have $F = 0$, and hence $\int f \in \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$. If $x_1 = 0, x_2 \neq 0, x_3 \neq 0$, then $Y = \frac{x_2 + x_3(u')^2}{u''} \partial - x_3 u'$ and $Z = \partial \circ \frac{1}{u''} \partial$. The equation $YF = 0$ has a one-dimensional (over $\mathcal{C}$) space of solution, spanned by $F = \sqrt{x_2 + x_3(u')^2}$. Hence, all elements $\int f \in \mathcal{F}_0(X)$ are obtained solving the equation

$$\frac{\delta f}{\delta u} = \alpha \partial \circ \frac{1}{u''} \partial \sqrt{x_2 + x_3(u')^2} = \alpha \left(\frac{x_3 u'}{\sqrt{x_2 + x_3(u')^2}} \right)' ,$$

for $\alpha \in \mathcal{C}$. Its solutions are of the form $\int f = -\alpha \sqrt{x_2 + x_3(u')^2} + k$, where $k \in \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$. Next, if $x_1 \neq 0, x_2 = x_3 = 0$, then $Y = x_1 \partial$ and $Z = 1$, so the equations (3.11) give $F \in \mathcal{C}$ and $\int f \in \mathcal{C} \int u + \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$. If $x_1 \neq 0, x_2 \neq 0, x_3 = 0$, then $Y = x_1 \partial^2 + x_2$ and $Z = \partial$. In this case, the first equation in (3.11) reads

$$x_1 F'' + x_2 F = 0 .$$

By Lemma 3.2(b), it must be $F \in \mathcal{F}$, and it is easy to see that the space of solutions is two-dimensional over $\mathcal{C}$, consisting of elements of the form

$$F = \alpha_+ e^{x_{12} x} + \alpha_- e^{-x_{12} x} ,$$

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with $\alpha_\pm \in \mathcal{C}$. Then, the second equation in (3.11) gives

$$\frac{\delta f}{\delta u} = \alpha_+ x_{12} e^{x_{12} x} - \alpha_- x_{12} e^{-x_{12} x} \,,$$

so that $\int f = \alpha_+ x_{12} \int e^{x_{12} x} u - \alpha_- x_{12} \int e^{-x_{12} x} u + k$, where $k \in \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$. Similarly, we consider the case $x_1 \neq 0, x_2 = 0, x_3 \neq 0$. In this case $Y = x_1 \partial \circ \frac{1}{u'}\partial + x_3 u'$ and $Z = \frac{1}{u'}\partial$. The first equation in (3.11) reads

$$x_1 \left(\frac{F'}{u'}\right)' + x_3 u' F = 0 \,.$$

By Lemma 3.2(c), we must have $F \in \mathcal{V}_0$ such that $F' = \frac{\partial F}{\partial u}$. It is easy to see that the space of solutions is two-dimensional over $\mathcal{C}$, and it consists of elements of the form

$$F = \alpha_+ e^{x_{13} u} + \alpha_- e^{-x_{13} u} \,,$$

with $\alpha_\pm \in \mathcal{C}$. Then, the second equation in (3.11) gives

$$\frac{\delta f}{\delta u} = \alpha_+ x_{13} e^{x_{13} u} - \alpha_- x_{13} e^{-x_{13} u} \,,$$

and its solutions for $\int f$ are of the form $\int f = \alpha_+ \int e^{x_{13} u} + \alpha_- \int e^{-x_{13} u} + k$, for $k \in \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$. Finally, we are left to consider the case when $x_1 \neq 0, x_2 \neq 0, x_3 \neq 0$. In this case $Y = x_1 \partial^2 \circ \frac{1}{u''}\partial + \frac{x_2 + x_3 (u')^2}{u''}\partial - x_3 u'$ and $Z = \partial \circ \frac{1}{u''}\partial$. The first equation in (3.11) reads

$$x_1 \left(\frac{F'}{u''}\right)'' + (x_2 + x_3 (u')^2)\frac{F'}{u''} - x_3 u' F = 0 \,.$$

By Lemma 3.2(a), the only solution of this equation is $F = 0$. But then the second equation in (3.11) gives $\int f \in \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$. $\qquad\square$

In the statement of Lemma 3.3 and further on in this section, we assume that $\mathcal{V}$ contains all the elements which appear in the statement, namely $e^{x_{12} x}$, $e^{x_{13} u}$, and $\sqrt{x_2 + x_3 (u')^2}$.

Next, for each element $\int f \in \mathcal{F}_0(X)$, we want to find an element $P \in \mathcal{H}(X)$ which is X-associated to it, and for each element $P \in \mathcal{H}_0(X)$, we want to find an element $\int f \in \mathcal{H}(X)$ which is X-associated to it. Recall, by Lemma 2.7, that if $\int f \xleftrightarrow{X} P$, then all elements in $\mathcal{H}(X)$ which are X-associated to $\int f$ are obtained adding to P an arbitrary element of $\mathcal{H}_0(X)$, and all elements in $\mathcal{F}(X)$ which are X-associated to P are obtained adding to $\int f$ an arbitrary element of $\mathcal{F}_0(X)$.

Lemma 3.4. *Let* $X = x_1 L_1 + x_2 L_2 + x_3 L_3$, *and let* $a_2, a_3, \gamma \in \mathcal{C}\backslash\{0\}$. *We have:*

(i) $\nexists P \in \mathcal{H}(X)$ *such that* $\int e^{\gamma x} u \xleftrightarrow{X} P$, *if* $x_3 \neq 0$;

$\int e^{\gamma x} u \xleftrightarrow{X} \frac{1}{\gamma}(x_1 \gamma^2 + x_2) e^{\gamma x}$, *if* $x_3 = 0$.

(ii) $\nexists P \in \mathcal{H}(X)$ *such that* $\int e^{\gamma u} \xleftrightarrow{X} P$, *if* $x_2 \neq 0$;

$\int e^{\gamma u} \xleftrightarrow{X} (x_1 \gamma^2 + x_3) e^{\gamma u} u'$, *if* $x_2 = 0$.

(iii) $\int u \xleftrightarrow{X} (x_2 x + x_3 u u')$.

(iv) $\int \sqrt{a_2 + a_3 (u')^2} \xleftrightarrow{X} -\left(x_1 \partial^2 \circ \frac{1}{u''} \partial + \frac{x_2 + x_3 (u')^2}{u''} \partial - x_3 u' \right) \sqrt{a_2 + a_3 (u')^2}$.

(v) $\int 0 \xleftrightarrow{X} 1$, *if* $x_2 \neq 0$;

$\nexists \int f \in \mathcal{F}(X)$ *such that* $\int f \xleftrightarrow{X} 1$, *if* $x_2 = 0, x_1 x_3 \neq 0$;

$\int \frac{1}{2 x_3 u'} \xleftrightarrow{X} 1$, *if* $x_1 = 0, x_2 = 0, x_3 \neq 0$;

$\int \frac{x u}{x_1} \xleftrightarrow{X} 1$, *if* $x_1 \neq 0, x_2 = 0, x_3 = 0$.

(vi) $\int 0 \xleftrightarrow{X} u'$, *if* $x_3 \neq 0$;

$\nexists \int f \in \mathcal{F}(X)$ *such that* $\int f \xleftrightarrow{X} u'$, *if* $x_3 = 0, x_1 x_2 \neq 0$;

$\int \frac{-(u')^2}{2 x_2} \xleftrightarrow{X} u'$, *if* $x_1 = 0, x_2 \neq 0, x_3 = 0$;

$\int \frac{u^2}{2 x_1} \xleftrightarrow{X} u'$, *if* $x_1 \neq 0, x_2 = 0, x_3 = 0$.

Proof. The condition $\int e^{\gamma x} u \xleftrightarrow{X} P$ means that, for some fractional decomposition $X = Y Z^{-1}$, there exists $F \in \mathcal{V}$ such that $P = Y F$ and $Z F = e^{\gamma x}$. Let us consider first the case $x_3 \neq 0$. In this case, by Lemma 3.1 a minimal fractional decomposition for X is (3.2) if $x_2 \neq 0$, and (3.4) if $x_2 = 0$. In the former case $Z = \partial \circ \frac{1}{u''} \partial$, hence the equation $Z F = e^{\gamma x}$ reads $\partial \frac{F'}{u''} = e^{\gamma x}$, namely

$$F' = \frac{1}{\gamma} e^{\gamma x} u'' + \alpha u'',$$

for some $\alpha \in \mathcal{C}$. This equation has no solutions since, integrating by parts, $\int \left(\frac{1}{\gamma} e^{\gamma x} u'' + \alpha u'' \right) = \gamma \int e^{\gamma x} u$, and this is not zero by (2.3). Similarly, in the case $x_2 = 0$ we have $Z = \frac{1}{u'} \partial$, hence the equation $Z F = e^{\gamma x}$ reads

$$F' = e^{\gamma x} u',$$

which has no solutions since, integrating by parts, $\int e^{\gamma x} u' = -\gamma \int e^{\gamma x} u \neq 0$. To conclude the proof of part (i), we consider the case $x_3 = 0$. By Lemma 3.1 a fractional decomposition for X is $X = YZ^{-1}$ given by (3.3). Hence, a solution $F \in \mathcal{V}$ to the equation $ZF = e^{\gamma x}$ is $F = \frac{1}{\gamma} e^{\gamma x}$, and in this case we have $P = YF = (x_1 \partial^2) \frac{1}{\gamma} e^{\gamma x} = \left(x_1 \gamma + \frac{x_2}{\gamma} \right) e^{\gamma x}$.

Next, let us prove part (ii). The condition $\int e^{\gamma u} \xleftrightarrow{X} P$ is equivalent to the existence of $F \in \mathcal{V}$ such that $P = YF$ and $ZF = \gamma e^{\gamma u}$, where $X = YZ^{-1}$. Let us consider first the case $x_2 \neq 0$. In this case, by Lemma 3.1 a minimal fractional decomposition $X = YZ^{-1}$ for X has $Z = \partial \circ \frac{1}{u''} \partial$ if $x_3 \neq 0$, and $Z = \partial$ if $x_3 = 0$. In both cases the equation $ZF = \gamma e^{\gamma u}$ would imply $\gamma e^{\gamma u} \in \partial \mathcal{V}$, which is not the case by (2.3). In the case $x_2 = 0$, a fractional decomposition for X is $X = YZ^{-1}$ given by (3.4). Hence, a solution $F \in \mathcal{V}$ to the equation $ZF = \gamma e^{\gamma u}$ is $F = e^{\gamma u}$, and in this case we have $P = YF = \left(x_1 \partial \circ \frac{1}{u'} \partial + x_3 u' \right) e^{\gamma u} = (x_1 \gamma^2 + x_3) e^{\gamma u} u'$.

For part (iii) it suffices to check, using the fractional decomposition (3.2), that $F = xu' - u \in \mathcal{V}$ is a solution of the equations

$$ZF = \partial \frac{F'}{u''} = \frac{\delta}{\delta u} \int u \,,$$
$$YF = \left(x_1 \partial^2 \circ \frac{1}{u''} \partial + \frac{x_2 + x_3 (u')^2}{u''} \partial - x_3 u' \right) F = x_2 x + x_3 u u' \,.$$

Similarly, for part (iv), letting $F = -\sqrt{a_2 + a_3 (u')^2} \in \mathcal{V}$, we have

$$ZF = \partial \frac{F'}{u''} = \frac{\delta}{\delta u} \int \sqrt{a_2 + a_3 (u')^2} \,,$$
$$YF = -\left(x_1 \partial^2 \circ \frac{1}{u''} \partial + \frac{x_2 + x_3 (u')^2}{u''} \partial - x_3 u' \right) \sqrt{a_2 + a_3 (u')^2} \,.$$

Next, let us prove part (v). For $x_2 \neq 0$, consider the fractional decomposition $X = YZ^{-1}$ given by (3.2). It is easy to check that, letting $F = \frac{u'}{x_2}$, we have

$$ZF = \partial \circ \frac{1}{u''} \partial \frac{u'}{x_2} = 0 \,,$$
$$YF = \left(x_1 \partial^2 \circ \frac{1}{u''} \partial + \frac{x_2 + x_3 (u')^2}{u''} \partial - x_3 u' \right) \frac{u'}{x_2} = 1 \,.$$

Hence, $\int 0 \xleftrightarrow{X} 1$, as we wanted. If $x_1 \neq 0, x_2 = 0, x_3 \neq 0$, a minimal fractional decomposition for X is $X = YZ^{-1}$ given by (3.4). Therefore the relation $\int f \xleftrightarrow{X} 1$ is equivalent to the existence of $F \in \mathcal{V}$ such that

(3.12) $$ZF = \frac{F'}{u'} = \frac{\delta f}{\delta u} \,, \quad YF = \left(x_1 \partial \circ \frac{1}{u'} \partial + x_3 u' \right) F = 1 \,.$$

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By Lemma 3.2(c), the second equation in (3.12) implies that $F \in \mathcal{V}_0$ is such that $F' = \frac{\partial F}{\partial u} u'$. In this case, the second equation in (3.12) reads

$$x_1 \partial \frac{\partial F}{\partial u} + x_3 F u' = 1 ,$$

which, by the commutation relation (2.1), is equivalent to

$$\left(x_1 \frac{\partial^2 F}{\partial u^2} + x_3 F \right) u' = 1 .$$

But obviously, the above equation is never satisfied. If $x_1 = 0, x_2 = 0, x_3 \neq 0$, it is easy to check that $F = \frac{1}{x_3 u'}$ solves

$$YF = x_3 u' \frac{1}{x_3 u'} = 1 , \quad ZF = \frac{1}{u'} \partial \frac{1}{x_3 u'} = \frac{\delta}{\delta u} \int \frac{1}{2 x_3 u'} ,$$

proving that $\int \frac{1}{2 x_3 u'} \overset{X}{\longleftrightarrow} 1$. Finally, $x_1 \neq 0, x_2 = 0, x_3 = 0$, a minimal fractional decomposition $X = YZ^{-1}$ is given by $Y = x_1 \partial$ and $Z = 1$. In this case, it is immediate to check that $F = \frac{x}{x_1}$ solves

$$YF = x_1 \partial \frac{x}{x_1} = 1 , \quad ZF = \frac{x}{x_1} = \frac{\delta}{\delta u} \int \frac{xu}{x_1} ,$$

proving that $\int \frac{xu}{x_1} \overset{X}{\longleftrightarrow} 1$.

We are left to prove part (vi). For $x_3 \neq 0$, consider the fractional decomposition $X = YZ^{-1}$ given by (3.2). It is easy to check that, letting $F = \frac{-1}{x_3}$, we have

$$ZF = \partial \circ \frac{1}{u''} \partial \frac{-1}{x_3} = 0 ,$$
$$YF = \left(x_1 \partial^2 \circ \frac{1}{u''} \partial + \frac{x_2 + x_3 (u')^2}{u''} \partial - x_3 u' \right) \frac{-1}{x_3} = u' .$$

Hence, $\int 0 \overset{X}{\longleftrightarrow} u'$, as we wanted. If $x_1 \neq 0, x_2 \neq 0, x_3 = 0$, the minimal fractional decomposition for X is $X = YZ^{-1}$ given by (3.3). Therefore the relation $\int f \overset{X}{\longleftrightarrow} u'$ is equivalent to the existence of $F \in \mathcal{V}$ such that

(3.13) $$ZF = F' = \frac{\delta f}{\delta u} , \quad YF = (x_1 \partial^2 + x_2) F = u' .$$

If $F \in \mathcal{V}_n$, for $n \geq 0$, we get, applying $\frac{\partial}{\partial u^{(n+2)}}$ to both sides of the second equation in (3.13), that $\frac{\partial F}{\partial u^{(n)}} = 0$. Hence, it must be $F \in \mathcal{F}$. But in this

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case, the second equation in (3.13) has clearly no solutions. If $x_1 = 0, x_2 \neq 0, x_3 = 0$, it is easy to check that $F = \frac{u'}{x_2}$ solves

$$YF = x_2 \frac{u'}{x_2} = u', \quad ZF = \partial \frac{u'}{x_2} = \frac{\delta}{\delta u} \int \frac{-(u')^2}{2x_2},$$

proving that $\int \frac{-(u')^2}{2x_2} \xleftrightarrow{X} u'$. Finally, if $x_1 \neq 0, x_2 = 0, x_3 = 0$, a minimal fractional decomposition $X = YZ^{-1}$ is given by $Y = x_1 \partial$ and $Z = 1$. In this case, it is immediate to check that $F = \frac{u}{x_1}$ solves

$$YF = x_1 \partial \frac{u}{x_1} = u', \quad ZF = \frac{u}{x_1} = \frac{\delta}{\delta u} \int \frac{u^2}{2x_1},$$

proving that $\int \frac{u^2}{2x_1} \xleftrightarrow{X} u'$. □

In order the check orthogonality conditions (2.17) for Liouville type integrable systems we will use the following results.

Lemma 3.5. (a) $(\mathcal{C}1)^\perp = \mathrm{Im}(\partial)$;

(b) $(\mathcal{C}u')^\perp = \mathrm{Im}(\frac{1}{u'}\partial)$;

(c) $\left(\mathrm{Span}_{\mathcal{C}}\{1, u'\}\right)^\perp = \mathrm{Im}(\partial \circ \frac{1}{u''}\partial)$;

(d) for $b_2, b_3 \in \mathcal{C}\backslash\{0\}$, we have

$$\left(\mathcal{C}\frac{\delta}{\delta u}\int \sqrt{b_2 + b_3(u')^2}\right)^\perp = \mathrm{Im}\left(\frac{b_2 + b_3(u')^2}{u''}\partial - b_3 u'\right).$$

Proof. Parts (a) and (b) are immediate. Let us prove part (c). It is immediate to check, integrating by parts, that $\int (\alpha + \beta u')\partial \frac{f'}{u''} = 0$ for every $\alpha, \beta \in \mathcal{C}$. Hence, $\mathrm{Im}(\partial \circ \frac{1}{u''}\partial) \subset \left(\mathrm{Span}_{\mathcal{C}}\{1, u'\}\right)^\perp$. On the other hand, if $f \in \left(\mathrm{Span}_{\mathcal{C}}\{1, u'\}\right)^\perp$, it must be

$$f = \partial g = \frac{1}{u'}\partial h \quad \text{for some } g, h \in \mathcal{V}.$$

But then $\partial h = u'\partial g = \partial(u'g) - u''g$, which implies $g = \frac{1}{u''}\partial(u'g - h)$. Hence,

$$f = \partial\frac{1}{u''}\partial(u'g - h) \in \mathrm{Im}(\partial \circ \frac{1}{u''}\partial).$$

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We are left to prove part (d). By definition of the variational derivative, we have

$$-\frac{1}{b_3}\frac{\delta}{\delta u}\int \sqrt{b_2 + b_3(u')^2} = \partial \frac{u'}{\sqrt{b_2 + b_3(u')^2}}\,.$$

The inclusion $\mathrm{Im}\left(\frac{b_2 + b_3(u')^2}{u''}\partial - b_3 u'\right) \subset \left(\mathcal{C}\partial\frac{u'}{\sqrt{b_2 + b_3(u')^2}}\right)^{\perp}$ follows by integration by parts, and the following straightforward identity

$$\left(\partial \circ \frac{b_2 + b_3(u')^2}{u''} + b_3 u'\right)\partial\frac{u'}{\sqrt{b_2 + b_3(u')^2}} = 0\,.$$

We are left to prove the opposite inclusion. If $f \in \left(\mathcal{C}\partial\frac{u'}{\sqrt{b_2 + b_3(u')^2}}\right)^{\perp}$, we have

$$f = \frac{\partial g}{\partial \frac{u'}{\sqrt{b_2 + b_3(u')^2}}} = \frac{(b_2 + b_3(u')^2)^{\frac{3}{2}}}{b_2 u''}\partial g\,,$$

for some $g \in \mathcal{V}$. Letting $g = \frac{h}{\sqrt{b_2 + b_3(u')^2}}$, we then get,

$$\begin{aligned}
f &= \frac{(b_2 + b_3(u')^2)^{\frac{3}{2}}}{b_2 u''}\partial\frac{h}{\sqrt{b_2 + b_3(u')^2}} \\
&= \frac{(b_2 + b_3(u')^2)^{\frac{3}{2}}}{b_2 u''}\left(-\frac{b_3 u' u''}{(b_2 + b_3(u')^2)^{\frac{3}{2}}}h + \frac{1}{\sqrt{b_2 + b_3(u')^2}}h'\right) \\
&= \left(\frac{b_2 + b_3(u')^2}{u''}\partial - b_3 u'\right)\frac{h}{b_2}\,.
\end{aligned}$$

$\square$

3.2 Integrability of the Lenard-Magri scheme in the case $b_1 = 0$

In this and the next two Sections we consider the case when $b_1 = 0$, for which we get integrable Lenard-Magri schemes of S-type, in the terminology introduced in Section 2.4, in the case $a_1 \neq 0$ (described in Section 3.3), and of C-type in the case $a_1 = 0$ (described in Section 3.4). In Sections 3.5 we will consider the remaining case, when $b_1 \neq 0$, for which we again get some integrable Lenard-Magri schemes of C-type.

According to Theorem 2.11, in order to apply successfully the Lenard-Magri scheme of integrability, we need to find finite sequences $\{P_n\}_{n=0}^N$, $\{\int h_n\}_{n=0}^N$, satisfying all the relations (2.16) and the orthogonality conditions (2.17). For $b_1 = 0$ we display below such sequences separately in

all possibilities for the coefficients a_2, a_3, b_2, b_3 being zero or non-zero, and a_1 arbitrary. (Note that, since we are assuming $b_1 = 0$, we don't need to consider the case $b_2 = b_3 = 0$.)

(i) $b_2 b_3 \neq 0, a_2 a_3 \neq 0$: $\int 0 \xleftrightarrow{H} 1 \xleftrightarrow{K} \int 0 \xleftrightarrow{H} u' \xleftrightarrow{K} \int \sqrt{b_2 + b_3 (u')^2}$.

(ii) $b_2 b_3 \neq 0, a_2 \neq 0, a_3 = 0$: $\int 0 \xleftrightarrow{H} 1 \xleftrightarrow{K} \int \sqrt{b_2 + b_3 (u')^2}$.

(iii) $b_2 b_3 \neq 0, a_2 = 0, a_3 \neq 0$: $\int 0 \xleftrightarrow{H} u' \xleftrightarrow{K} \int \sqrt{b_2 + b_3 (u')^2}$.

(iv) $b_2 b_3 \neq 0, a_2 = a_3 = 0$: $\int 0 \xleftrightarrow{H} 0 \xleftrightarrow{K} \int \sqrt{b_2 + b_3 (u')^2}$.

(v) $b_2 \neq 0, b_3 = 0, a_2 a_3 \neq 0$: $\int 0 \xleftrightarrow{H} 1 \xleftrightarrow{K} \int 0 \xleftrightarrow{H} u' \xleftrightarrow{K} \int \frac{-(u')^2}{2b_2}$.

(vi) $b_2 \neq 0, b_3 = 0, a_2 \neq 0, a_3 = 0$: $\int 0 \xleftrightarrow{H} 1 \xleftrightarrow{K} \int 0$.

(vii) $b_2 \neq 0, b_3 = 0, a_2 = 0, a_3 \neq 0$: $\int 0 \xleftrightarrow{H} u' \xleftrightarrow{K} \int \frac{-(u')^2}{2b_2}$.

(viii) $b_2 \neq 0, b_3 = 0, a_2 = a_3 = 0$: $\int 0 \xleftrightarrow{H} 0 \xleftrightarrow{K} \int 0$.

(ix) $b_2 = 0, b_3 \neq 0, a_2 a_3 \neq 0$: $\int 0 \xleftrightarrow{H} u' \xleftrightarrow{K} \int 0 \xleftrightarrow{H} 1 \xleftrightarrow{K} \int \frac{1}{2b_3 u'}$.

(x) $b_2 = 0, b_3 \neq 0, a_2 \neq 0, a_3 = 0$: $\int 0 \xleftrightarrow{H} 1 \xleftrightarrow{K} \int \frac{1}{2b_3 u'}$.

(xi) $b_2 = 0, b_3 \neq 0, a_2 = 0, a_3 \neq 0$: $\int 0 \xleftrightarrow{H} u' \xleftrightarrow{K} \int 0$.

(xii) $b_2 = 0, b_3 \neq 0, a_2 = a_3 = 0$: $\int 0 \xleftrightarrow{H} 0 \xleftrightarrow{K} \int 0$.

All the above H- and K-association relations hold due to Lemmas 3.3 and 3.4. Moreover, using Lemmas 3.1 and 3.5 we check that both orthogonality conditions (2.17) hold. Hence, by Theorem 2.11 all the above sequences can be continued indefinitely to an infinite sequence

$$\int 0 \xleftrightarrow{H} P_0 \xleftrightarrow{K} \int h_0 \xleftrightarrow{H} P_1 \xleftrightarrow{K} \int h_1 \xleftrightarrow{H} \ldots .$$

Note that, by Lemma 2.7, at each step the subsequent term is unique up to a linear combinations of the previous steps.

Next, we want to discuss integrability of the corresponding hierarchies of Hamiltonian equations $\frac{du}{dt_n} = P_n$, $n \in \mathbb{Z}_+$. Namely, we need to see when the vector spaces $\mathrm{Span}_{\mathcal{C}}\{\int h_n\} \subset \mathcal{V}/\partial\mathcal{V}$ and $\mathrm{Span}_{\mathcal{C}}\{P_n\} \subset \mathcal{V}$ are infinite dimensional.

First, we consider the cases (vi), (viii), (xi) and (xii), where we show that integrability does not occur (regardless of a_1 being zero or non-zero) since the Lenard-Magri scheme repeats itself. In case (vi), by Lemmas 3.3 and 3.4 we have $\mathcal{H}_0(H) = \mathcal{C}$ and, for every $\alpha \in \mathcal{C}$, $\{\int f \in \mathcal{F}(K) \mid \int f \xleftrightarrow{K} \alpha\} = \mathcal{F}_0(K) = \text{Ker}\left(\frac{\delta}{\delta u}\right)$. Hence, any infinite sequence extending the given finite one will have $\int h_n \in \text{Ker}\left(\frac{\delta}{\delta u}\right)$ and $P_n \in \mathcal{C}$, for every $n \in \mathbb{Z}_+$. Similarly, in case (xi) we have $\mathcal{H}_0(H) = \mathcal{C}u'$ and, for every $\alpha u' \in \mathcal{C}u'$, $\{\int f \in \mathcal{F}(K) \mid \int f \xleftrightarrow{K} \alpha u'\} = \mathcal{F}_0(K) = \text{Ker}\left(\frac{\delta}{\delta u}\right)$. Hence, any infinite sequence extending the given finite one will have $\int h_n \in \text{Ker}\left(\frac{\delta}{\delta u}\right)$ and $P_n \in \mathcal{C}u'$, for every $n \in \mathbb{Z}_+$. In cases (viii) and (xii) we have $\mathcal{H}_0(H) = 0$ and $\mathcal{F}_0(K) = \text{Ker}\left(\frac{\delta}{\delta u}\right)$. Hence, $\int h_n \in \text{Ker}\left(\frac{\delta}{\delta u}\right)$ and $P_n = 0$, for every $n \in \mathbb{Z}_+$. In conclusion, in all these cases $\text{Span}_\mathcal{C}\{P_n\}$ is finite dimensional, and integrability does not occur.

For the remaining 8 cases, we prove in Section 3.3 that when $a_1 \neq 0$ we get some integrable Lenard-Magri scheme of S-type, and we prove in Section 3.4 that when $a_1 = 0$ we get some integrable Lenard-Magri scheme of C-type (in the terminology of Section 2.4).

3.3 Integrable Lenard-Magri schemes of S-type: $b_1 = 0$ and $a_1 \neq 0$

Cases $(i), (ii), (iii), (iv)$

In all the sequences (i)-(iv), after one or two steps, we arrive at (after shifting indices) $\int h_{-1} = \int \sqrt{b_2 + b_3(u')^2}$. The next term in the sequence, which we denote P_0, is obtained by solving the following equations for F and P_0 in $\mathcal{V}$:

$$BF = \partial \circ \frac{1}{u''} \partial F = \frac{\delta}{\delta u} \int\!\!\int \sqrt{b_2 + b_3(u')^2},$$
$$P_0 = AF = \left(a_1 \partial^2 \circ \frac{1}{u''} \partial + \frac{a_2 + a_3(u')^2}{u''} \partial - a_3 u'\right) F.$$

It is easy to check that a solution is given by $F = -\sqrt{b_2 + b_3(u')^2}$, and

(3.14)
$$P_0 = -\left(\frac{a_1 b_3 u'}{\sqrt{b_2 + b_3(u')^2}}\right)'' + \frac{(a_3 b_2 - a_2 b_3)}{\sqrt{b_2 + b_3(u')^2}}$$
$$= -\frac{a_1 b_2 b_3 u'''}{(b_2 + b_3(u')^2)^{\frac{3}{2}}} + 3\frac{a_1 b_2 b_3^2 u'(u'')^2}{(b_2 + b_3(u')^2)^{\frac{5}{2}}} + \frac{(a_3 b_2 - a_2 b_3)}{\sqrt{b_2 + b_3(u')^2}}.$$

The above computation works regardless whether a_1 is zero or not. But to prove that the Lenard-Magri scheme is integrable of S-type we need to assume $a_1 \neq 0$, in which case $\text{dord}(P_0) = 3$ is greater than $\max\{\text{dord}(A) - |H| +$

$|K|$, $\mathrm{dord}(B)+|K|$, $\mathrm{dord}(C)$, $\mathrm{dord}(D)+|K|\}$, which is less than or equal to 2 (for all the cases (i)-(iv)). By Proposition 2.15 it follows that $\mathrm{dord}(P_n) = 2n+3$ for every $n \geq 0$. In particular, the elements P_n, $n \in \mathbb{Z}_+$, are linearly independent. Therefore, by Theorem 2.11 and Remark 2.13, each Hamiltonian PDE $\frac{du}{dt} = P_n$, $n \in \mathbb{Z}_+$, is integrable, associated to an integrable Lenard-Magri scheme of S-type. (Note that, since $\mathrm{Ker}(B^*) \cap \mathrm{Ker}(D^*) = \mathcal{C} \oplus \mathcal{C}u' \neq 0$, we cannot conclude using Theorem 2.11 that $[P_m, P_n] = 0$ for every $m, n \in \mathbb{Z}_+$, and therefore that the sequence of equations $\frac{du}{dt_n} = P_n$, $n \in \mathbb{Z}_+$, form a compatible hierarchy. We conjecture, though, that this is the case.)

After rescaling x and t appropriately in the equation $\frac{du}{dt} = P_0$, we conclude that the following bi-Hamiltonian equation is integrable, associated to an integrable Lenard-Magri scheme of S-type:

$$(3.15) \qquad \frac{du}{dt} = \frac{u'''}{(1+(u')^2)^{\frac{3}{2}}} - 3\frac{u'(u'')^2}{(1+(u')^2)^{\frac{5}{2}}} + \frac{\alpha}{(1+(u')^2)^{\frac{1}{2}}} \ , \quad \alpha \in \mathcal{C}\,.$$

This is an equation of the form [MSS90, eq.(41.5)] with $a_3 = (1+(u')^2)^{\frac{1}{2}}$ (but in fact this particular a_3 does not appear in their list).

Cases $(v), (vii)$

In the sequences (v) and (vii), after one or two steps, we arrive at (after shifting indices) $\int h_{-1} = \int \frac{-(u')^2}{2b_2}$. The next term in the sequence, which we denote P_0, is obtained by solving the following equations for F and P_0 in $\mathcal{V}$:

$$BF = \partial \circ \frac{1}{u''}\partial F = \frac{\delta}{\delta u}\int\int \frac{-(u')^2}{2b_2}\,,$$
$$P_0 = AF = \left(a_1\partial^2 \circ \frac{1}{u''}\partial + \frac{a_2 + a_3(u')^2}{u''}\partial - a_3u'\right)F\,.$$

It is easy to check that a solution is given by $F = \frac{(u')^2}{2b_2}$, and

$$(3.16) \qquad\qquad P_0 = \frac{a_1}{b_2}u''' + \frac{a_2}{b_2}u' + \frac{a_3}{2b_2}(u')^3\,.$$

As before, if $a_1 \neq 0$, $\mathrm{dord}(P_0) = 3$ is greater than $\max\{\mathrm{dord}(A) - |H| + |K|, \mathrm{dord}(B) + |K|, \mathrm{dord}(C), \mathrm{dord}(D) + |K|\}$, which is at most 2. By Proposition 2.15 it follows that $\mathrm{dord}(P_n) = 2n+3$ for every $n \geq 0$. In particular, the elements P_n, $n \in \mathbb{Z}_+$, are linearly independent. Therefore, by Theorem 2.11 and Remark 2.13, each Hamiltonian PDE $\frac{du}{dt} = P_n$, $n \in \mathbb{Z}_+$, is integrable, associated to an integrable Lenard-Magri scheme of S-type.

After rescaling x and t appropriately in the equation $\frac{du}{dt} = P_0$, we conclude that the following bi-Hamiltonian equation is integrable, associated to an integrable Lenard-Magri scheme of S-type:

$$(3.17) \qquad \frac{du}{dt} = u''' + \epsilon u' + \alpha (u')^3 \,,$$

where ϵ is 1 (in case (v)) or 0 (in case (vii)) and $\alpha \in \mathcal{C}$. By a Galilean transformation we can make $\epsilon = 0$. The resulting equation is called the potential modified KdV equation (equation (4.11) in the list of [MSS90]).

Cases $(ix), (x)$

In all the sequences (ix) and (x), after one or two steps, we arrive at (after shifting indices) $\int h_{-1} = \int \frac{1}{2b_3 u'}$. The next term in the sequence, which we denote P_0, is obtained by solving the following equations for F and P_0 in $\mathcal{V}$:

$$BF = \partial \circ \frac{1}{u''} \partial F = \frac{\delta}{\delta u} \int \int \frac{1}{2b_3 u'} \,,$$
$$P_0 = AF = \left(a_1 \partial^2 \circ \frac{1}{u''} \partial + \frac{a_2 + a_3 (u')^2}{u''} \partial - a_3 u' \right) F \,.$$

It is easy to check that a solution is given by $F = \frac{-1}{2b_3 u'}$, and

$$(3.18) \qquad P_0 = -\frac{a_1}{b_3} \frac{u'''}{(u')^3} + \frac{3a_1}{b_3} \frac{(u'')^2}{(u')^4} + \frac{a_2}{2b_3} \frac{1}{(u')^2} + \frac{a_3}{b_3} \,.$$

If $a_1 \neq 0$, we have $\mathrm{dord}(P_0) = 3$, which is greater than $\max\{\mathrm{dord}(A) - |H| + |K|, \mathrm{dord}(B) + |K|, \mathrm{dord}(C), \mathrm{dord}(D) + |K|\}$, which is at most 2. By Proposition 2.15 it follows that $\mathrm{dord}(P_n) = 2n + 3$ for every $n \geq 0$. In particular, the elements P_n, $n \in \mathbb{Z}_+$, are linearly independent. Therefore, by Theorem 2.11 and Remark 2.13, each Hamiltonian PDE $\frac{du}{dt} = P_n$, $n \in \mathbb{Z}_+$, is integrable, associated to an integrable Lenard-Magri scheme of S-type.

After rescaling x and t appropriately in the equation $\frac{du}{dt} = P_0$, we conclude that the following bi-Hamiltonian equation is integrable, associated to an integrable Lenard-Magri scheme of S-type:

$$(3.19) \qquad \frac{du}{dt} = \frac{u'''}{(u')^3} - 3\frac{(u'')^2}{(u')^4} + \frac{1}{(u')^2} + \alpha \,, \quad \alpha \in \mathcal{C} \,.$$

As explained in [MSS90], by a point transformation one can reduce this equation to an equation of the form (4.1.4) in their list.

Remark 3.6. Note that equation $\frac{du}{dt} = P_0$ with P_0 given by (3.18) is transformed, by the hodograph transformation $x \mapsto u$, $u \mapsto -x$, to the equation with P_0 given by (3.16), after exchanging a_2 and b_2 with a_3 and b_3 respectively. Equivalently, equation (3.19) can be transformed to equation (3.17) up to rescaling of x and t.

3.4 Integrable Lenard-Magri schemes of C-type with $a_1 = b_1 = 0$

Cases $(i), (ii), (iii), (iv)$

As pointed out above, in all the sequences (i)-(iv) we arrive, after one or two steps (and after shifting indices), at $\int h_{-1} = \int \sqrt{b_2 + b_3(u')^2}$. In the case $a_1 = 0$ we can actually find an explicit solution for the sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$ and $\{P_n\}_{n \in \mathbb{Z}_+}$ satisfying the recursive formulas

$$(3.20) \qquad P_n \xleftrightarrow{K} \int h_n \xleftrightarrow{H} P_{n+1} , \quad n \in \mathbb{Z}_+ .$$

It is given by ($n \geq 0$):

$$(3.21) \qquad \begin{aligned} P_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{(2n-1-2k)!!}{(2n-2k)!!} \frac{\Delta^{n+1-k} a_3^k}{b_3^n} \frac{-u'}{(b_2 + b_3(u')^2)^{\frac{1}{2}+n-k}} , \\ h_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{(2n-1-2k)!!}{(2n-2k+2)!!} \frac{\Delta^{n+1-k} a_3^k}{b_3^{n+1}} \frac{-u'}{(b_2 + b_3(u')^2)^{\frac{1}{2}+n-k}} , \end{aligned}$$

where $\Delta = a_2 b_3 - a_3 b_2$, which is non-zero unless the operators H and K are proportional. Here and further we let $(-1)!! = 1$.

First, note that for $n = 0$, the above expression for P_0 is the same as the one in (3.14) with $a_1 = 0$. Hence $\int h_{-1} \xleftrightarrow{H} P_0$. Next, we check that indeed the sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$ and $\{P_n\}_{n \in \mathbb{Z}_+}$ solve the recursive relations (3.20). For this, we fix the fractional decompositions $H = AB^{-1}$ and $K = CD^{-1}$ given by

$$A = \frac{a_2 + a_3(u')^2}{u''} \partial - a_3 u' , \quad C = \frac{b_2 + b_3(u')^2}{u''} \partial - b_3 u' , B = D = \partial \circ \frac{1}{u''} \partial .$$

Since $B = D$, the relations (3.20) hold if there exists an element F_n such that

$$(3.22) \qquad CF_n = P_n , \quad BF_n = \frac{\delta h_n}{\delta u} , \quad P_{n+1} = AF_n .$$

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A solution F_n to equations (3.22) is given by $F_n = -h_n$. Since h_n depends only on u', we have

$$B(-h_n) = -\partial \circ \frac{1}{u''} \partial h_n = (-\partial) \frac{\partial h_n}{\partial u'} = \frac{\delta h_n}{\delta u} \, ,$$

hence $F_n = -h_n$ satisfies the second equation in (3.22). The first and third equations in (3.22) can be easily checked using the following straightforward identities ($m \in \mathbb{Z}_+$):

$$C \frac{1}{(b_2 + b_3(u')^2)^{\frac{m}{2}}} = \frac{-b_3 m u'}{(b_2 + b_3(u')^2)^{\frac{m}{2}}} \, ,$$

$$A \frac{1}{(b_2 + b_3(u')^2)^{\frac{m}{2}}} = m\Delta \frac{-u'}{(b_2 + b_3(u')^2)^{\frac{m}{2}+1}} + (m+1)a_3 \frac{-u'}{(b_2 + b_3(u')^2)^{\frac{m}{2}}} \, .$$

If $\Delta \neq 0$, all the elements P_n are linearly independent, therefore, by Theorem 2.11 and Remark 2.13, each Hamiltonian PDE $\frac{du}{dt} = P_n$, $n \in \mathbb{Z}_+$, is integrable, associated to an integrable Lenard-Magri scheme of C-type.

Cases $(v), (vii)$

In the sequences (v) and (vii) we arrive, after one or two steps (and after shifting indices), at $\int h_{-1} = \int \frac{-(u')^2}{2b_2}$. In the case $a_1 = 0$ we can actually find an explicit solution for the sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$ and $\{P_n\}_{n \in \mathbb{Z}_+}$ satisfying the recursive formulas (3.20). It is given by ($n \in \mathbb{Z}_+$):

(3.23)
$$P_{n-1} = \sum_{k=0}^{n} \binom{n}{k} \frac{(2k-1)!!}{(2k)!!} \frac{a_2^{n-k} a_3^k}{b_2^n} (u')^{2k+1} \, ,$$
$$h_{n-1} = -\sum_{k=0}^{n} \binom{n}{k} \frac{(2k-1)!!}{(2k+2)!!} \frac{a_2^{n-k} a_3^k}{b_2^n} (u')^{2k+2} \, .$$

First, note that the above expression for P_0 is the same as the one in (3.16) with $a_1 = 0$. Hence $\int h_{-1} \xleftrightarrow{H} P_0$. Next, we check that indeed the sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$ and $\{P_n\}_{n \in \mathbb{Z}_+}$ solve the recursive relations (3.20). For this, we fix the fractional decompositions $H = AB^{-1}$ and $K = CD^{-1}$ given by

$$A = \frac{a_2 + a_3(u')^2}{u''} \partial - a_3 u' \, , \quad B = \partial \circ \frac{1}{u''} \partial \, , \quad C = b_2 \, , \quad D = \partial.$$

The relations (3.20) hold if there exist elements $F_n, G_n \in \mathcal{V}$ such that

(3.24) $\quad CF_n = P_n \, , \quad DF_n = \frac{\delta h_n}{\delta u} \, , \quad BG_n = \frac{\delta h_n}{\delta u} \, , \quad P_{n+1} = AG_n \, .$

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Solutions F_n, G_n to equations (3.24) are given by $F_n = \frac{1}{b_2} P_n$ and $G_n = -h_n$. The first and third equations in (3.24) are immediate. The third equation follows from the immediate identity $\frac{P_n}{b_2} = -\frac{\partial h_n}{\partial u'}$. Finally, the fourth identity in (3.24) is easily checked using the Tartaglia-Pascal triangle.

Clearly, if $a_3 \neq 0$, all the elements P_n are linearly independent, therefore, by Theorem 2.11 and Remark 2.13, each Hamiltonian PDE $\frac{du}{dt} = P_n$, $n \in \mathbb{Z}_+$, is integrable, associated to an integrable Lenard-Magri scheme of C-type.

Cases $(ix), (x)$

In the sequences (ix) and (x) we arrive, after one or two steps (and shift of indices), at $\int h_{-1} = \int \frac{1}{2b_3 u'}$. If $a_1 = 0$ we can find an explicit solution for the sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$ and $\{P_n\}_{n \in \mathbb{Z}_+}$ satisfying the recursive formulas (3.20). It is given by $(n \in \mathbb{Z}_+)$:

(3.25)
$$P_{n-1} = \sum_{k=0}^{n} \binom{n}{k} \frac{(2k-1)!!}{(2k)!!} \frac{a_3^{n-k} a_2^k}{b_3^n} \frac{1}{(u')^{2k}},$$
$$h_{n-1} = \sum_{k=0}^{n} \binom{n}{k} \frac{(2k-1)!!}{(2k+2)!!} \frac{a_3^{n-k} a_2^k}{b_3^{n+1}} \frac{1}{(u')^{2k+1}}.$$

First, note that the above expression for P_0 is the same as the one in (3.18) with $a_1 = 0$. Hence $\int h_{-1} \xleftrightarrow{H} P_0$. Next, we check that indeed the sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$ and $\{P_n\}_{n \in \mathbb{Z}_+}$ solve the recursive relations (3.20). For this, we fix the fractional decompositions $H = AB^{-1}$ and $K = CD^{-1}$ given by

$$A = \frac{a_2 + a_3(u')^2}{u''} \partial - a_3 u' \,, \quad B = \partial \circ \frac{1}{u''} \partial \,, \quad C = b_3 u' \,, \quad D = \frac{1}{u'} \partial \,.$$

Since equations (3.24) hold with $F_n = \frac{1}{b_3 u'} P_n$ and $G_n = -h_n$ (a fact that can be easily checked directly), it follows that the recursive relations (3.20) hold.

Clearly, if $a_2 \neq 0$, all the elements P_n are linearly independent, therefore, by Theorem 2.11 and Remark 2.13, each Hamiltonian PDE $\frac{du}{dt} = P_n$, $n \in \mathbb{Z}_+$, is integrable, associated to an integrable Lenard-Magri scheme of C-type.

3.5   Integrable Lenard-Magri scheme of C-type with $b_1 \neq 0$

As we did in the previous sections, we study here the integrability of the Lenard-Magri scheme when $b_1 \neq 0$. We will consider separately the various

cases, depending on the parameters b_2, b_3, a_2, a_3 being zero or non-zero.

Case 1: $b_2 b_3 \neq 0$

Let us consider first the case when b_2 and b_3 are both non-zero. If $\int h_0 \in \mathcal{V}/\partial\mathcal{V}$ and $P_0 \in \mathcal{V}$ satisfy the relations $\int 0 \stackrel{H}{\longleftrightarrow} P_0 \stackrel{K}{\longleftrightarrow} \int h_0$, then, by Lemmas 3.3 and 3.4, we necessarily have $P_0 \in \mathcal{C} \oplus \mathcal{C}u'$ and $\frac{\delta h_0}{\delta u} = 0$. Hence, any infinite sequence extending the given finite one will have $\int h_n \in \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$ and $P_n \in \mathcal{C} \oplus \mathcal{C}u'$, for every $n \in \mathbb{Z}_+$. In other words, the Lenard-Magri scheme repeats itself and integrability does not occur.

Case 2: $b_2 \neq 0, b_3 = 0, a_3 = 0$

In the case when $b_1 b_2 \neq 0$, $b_3 = 0$ and $a_3 = 0$, we can find explicitly all possible solutions for the sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$ and $\{P_n\}_{n \in \mathbb{Z}_+}$ satisfying the Lenard-Magri recursive relations (2.18).

In order to describe such solutions, we need to introduce some polynomials. We let $p_n(x; A, \epsilon)$, $q_n(x; A, \epsilon)$, $n \in \mathbb{Z}_+$, be the sequences of polynomials, depending on the 2×2 matrix $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$, and on the sequence of constant parameters $\epsilon = (\epsilon_0, \epsilon_1, \dots)$, defined by the following recursive relations: $p_0(x; A, \epsilon) = 0$, and

(3.26)
$$p_{n+1}(x; A, \epsilon) = \frac{a_1}{b_1} p_n(x; A, \epsilon) + \frac{a_2 b_1 - a_1 b_2}{b_1^2} q_n(x; A, \epsilon)$$
$$\left(\frac{d^2}{dx^2} + 2b_{12}\frac{d}{dx}\right) q_n(x; A, \epsilon) = p_n(x; A, \epsilon).$$

Here and further, as before, we use the notation (3.9) with x_i and x_j replaced by b_i and b_j. It is easy to see that, if $p(x)$ is a polynomial of degree n, then a solution $q(x)$ of the differential equation $q''(x) + 2b_{12}q'(x) = p(x)$ is a polynomial of degree $n + 1$, defined uniquely up to an additive constant ϵ_0. Hence, at each step in the recursion (3.26), the resulting polynomial $p_{n+1}(x)$ depends on the previous step $p_n(x)$ and on the choice of a constant parameter ϵ_{n+1}.

With the above notation, all sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$, $\{P_n\}_{n \in \mathbb{Z}_+}$, satisfying

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the Lenard-Magri recursive relations (2.18), are as follows:

$$P_n = p_n(x; A, \epsilon^+)e^{b_{12}x} + p_n(-x; A, \epsilon^-)e^{-b_{12}x} + a_2\delta_n \, ,$$

(3.27)
$$h_n = \frac{1}{b_1}\Big(q'_n(x; A, \epsilon^+) + b_{12}q_n(x; A, \epsilon^+)\Big)e^{b_{12}x}u$$

$$- \frac{1}{b_1}\Big(q'_n(-x; A, \epsilon^-) + b_{12}q_n(-x; A, \epsilon^-)\Big)e^{-b_{12}x}u \, ,$$

where $\epsilon^\pm = (\epsilon_0^\pm, \epsilon_1^\pm, \dots)$ and $\delta = (\delta_0, \delta_1, \dots)$ are arbitrary sequences of constant parameters.

It is not hard to check that the sequences $\{\int h_n\}_{n\in\mathbb{Z}_+}$ and $\{P_n\}_{n\in\mathbb{Z}_+}$ indeed solve the recursive relations (2.18), and any solution of the recursive relations (2.18) is obtained in this way. To conclude, we observe that, since $\Delta = a_2 b_1 - a_1 b_2$ is non-zero (unless the operators H and K are proportional), all the elements P_n are linearly independent, therefore, by Theorem 2.11 and Remark 2.13, each Hamiltonian PDE $\frac{du}{dt} = P_n$, $n \in \mathbb{Z}_+$, is integrable, associated to an integrable Lenard-Magri scheme of C-type.

Case 3: $b_2 \neq 0, b_3 = 0, a_3 \neq 0$

In the case when $b_1 b_2 \neq 0$, $b_3 = 0$ and $a_3 \neq 0$, we have by Lemma 3.3(b) that $\mathcal{H}_0(H) = a_2\mathcal{C} \oplus \mathcal{C}u'$. On the other hand, by Lemma 3.4(vi) there is no element $\int f \in \mathcal{F}(K)$ such that $\int f \xleftrightarrow{K} u'$. Similarly, by Lemma 3.3(a) we have $\mathcal{F}_0(K) = \mathcal{C}\int e^{b_{12}x}u + \mathcal{C}\int e^{-b_{12}x}u + \text{Ker}\left(\frac{\delta}{\delta u}\right)$. On the other hand, by Lemma 3.4(i) there is no element $P \in \mathcal{F}(H)$ such that $\int e^{\pm b_{12}x}u \xleftrightarrow{H} P$.

In conclusion, the Lenard-Magri recursion scheme, in this case, cannot be applied, since the following finite sequences cannot be extended to infinite sequences satisfying the conditions (2.18):

$$\int 0 \xleftrightarrow{H} u' \xleftrightarrow{K} \not\exists \int f \, , \quad \int 0 \xleftrightarrow{H} 0 \xleftrightarrow{K} \int e^{\pm b_{12}x}u \xleftrightarrow{H} \not\exists P \, .$$

Case 4: $b_2 = 0, b_3 \neq 0, a_2 = 0$

In the case when $b_1 b_3 \neq 0$, $b_2 = 0$ and $a_2 = 0$, we can find explicitly all possible solutions for the sequences $\{\int h_n\}_{n\in\mathbb{Z}_+}$ and $\{P_n\}_{n\in\mathbb{Z}_+}$ satisfying the

Lenard-Magri recursive relations (2.18). They are as follows:

$$P_n = p_n(u; A, \epsilon^+)e^{b_{13}u} + p_n(-u; A, \epsilon^-)e^{-b_{13}u} + a_3\delta_n u',$$

(3.28)
$$\frac{\delta h_n}{\delta u} = \frac{1}{b_1}\Big(q'_n(u; A, \epsilon^+) + b_{13}q_n(u; A, \epsilon^+)\Big)e^{b_{13}u}$$

$$- \frac{1}{b_1}\Big(q'_n(-u; A, \epsilon^-) + b_{13}q_n(-u; A, \epsilon^-)\Big)e^{-b_{12}u},$$

where $p_n(u; A, \epsilon^+)$ and $q_n(u; A, \epsilon^+)$ are the polynomials defined in (3.26), depending on the matrix $A = \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix}$, and on the sequences of constant parameters $\epsilon^\pm = (\epsilon_0^\pm, \epsilon_1^\pm, \dots)$ and $\delta = (\delta_0, \delta_1, \dots)$.

It is not hard to check, as in case 1, that the sequences $\{\int h_n\}_{n\in\mathbb{Z}_+}$ and $\{P_n\}_{n\in\mathbb{Z}_+}$ solve the recursive relations (2.18), and any solution of the recursive relations (2.18) is obtained in this way. We also observe that, since $\Delta = a_3b_1 - a_1b_3$ is non-zero (unless the operators H and K are proportional), all the elements P_n are linearly independent, therefore, by Theorem 2.11 and Remark 2.13, each Hamiltonian PDE $\frac{du}{dt} = P_n$, $n \in \mathbb{Z}_+$, is integrable, associated to an integrable Lenard-Magri scheme of C-type.

Case 5: $b_2 = 0, b_3 \neq 0, a_2 \neq 0$

In the case when $b_1b_3 \neq 0$, $b_2 = 0$ and $a_3 = 0$, we have by Lemma 3.3(b) that $\mathcal{H}_0(H) = \mathcal{C} \oplus a_3\mathcal{C}u'$. On the other hand, by Lemma 3.4(v) there is no element $\int f \in \mathcal{F}(K)$ such that $\int f \xrightarrow{K} 1$. Similarly, by Lemma 3.3(a) we have $\mathcal{F}_0(K) = \mathcal{C}\int e^{b_{13}u} + \mathcal{C}\int e^{-b_{13}u} + \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$. On the other hand, by Lemma 3.4(ii) there is no element $P \in \mathcal{F}(H)$ such that $\int e^{\pm b_{13}x}u \xrightarrow{H} P$.

In conclusion, the Lenard-Magri recursion scheme, in this case, cannot be applied, since the following finite sequences cannot be extended to infinite sequences satisfying the conditions (2.18):

$$\int 0 \xrightarrow{H} 1 \xrightarrow{K} \not\exists \int f, \quad \int 0 \xrightarrow{H} 0 \xrightarrow{K} \int e^{\pm b_{13}u} \xrightarrow{H} \not\exists P.$$

Case 6: $b_2 = b_3 = 0$

In the case when $b_1 \neq 0$, which we set equal to 1, and $b_2 = b_3 = 0$, we have different possibilities according to the constants a_2 and a_3 being zero or not.

If $a_2a_3 \neq 0$, the Lenard-Magri recursion scheme cannot be applied. Indeed, by Lemma 3.3 we have $\mathcal{H}_0(H) = \mathcal{C} \oplus \mathcal{C}u'$ and $\mathcal{F}_0(K) = \mathcal{C}\int u \oplus \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$,

and, whichever way we start the finite sequences $\{\int h_n\}_{n=0}^N$, $\{P_n\}_{n=0}^N$ as in (2.16), there is no way to extend them to non-trivial infinite sequences:

$$\textstyle\int 0 \xleftrightarrow{H} 1 \xleftrightarrow{K} \int xu \xleftrightarrow{H} \not\exists P_1\,,$$

$$\textstyle\int 0 \xleftrightarrow{H} u' \xleftrightarrow{K} \int \frac{1}{2}u^2 \xleftrightarrow{H} \not\exists P_1\,,$$

$$\textstyle\int 0 \xleftrightarrow{H} 0 \xleftrightarrow{K} \int u \xleftrightarrow{H} a_2 x + a_3 u u' \xleftrightarrow{K} \not\exists \int h_1\,.$$

Next, we consider the cases when exactly one element between a_2 and $a_3 = 0$ is zero. Recall the sequence of polynomials $p_n(x; A, \epsilon)$ defined by the recursive equations (3.26). In the case $b_1 = 1, b_2 = 0$, such equations reduce to $p_0(x; a_1, a_2, \epsilon) = 0$ and

$$(3.29) \qquad p_{n+1}(x; a_1, a_2, \epsilon) = \left(\frac{a_1}{b_1} + \frac{a_2}{b_1} \left(\frac{d}{dx} \right)^{-2} \right) p_n(x; a_1, a_2, \epsilon)\,.$$

Here $\left(\frac{d}{dx} \right)^{-2})$ means integrating twice with respect to x, which is defined uniquely up to adding a linear term $\epsilon_{2n} + \epsilon_{2n+1} x$. In particular, at each step the degree increases by two.

In the case $a_2 \neq 0, a_3 = 0$, it is not hard to prove that all the sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$, $\{P_n\}_{n \in \mathbb{Z}_+}$, satisfying the Lenard-Magri recursive relations (2.18), are as follows:

$$(3.30) \qquad P_n = p_n(x; a_1, a_2, \epsilon) + \delta_n\,, \quad \int h_n = \int \left(\frac{d}{dx} \right)^{-1} p_n(x; a_1, a_2, \epsilon) u\,,$$

where $\epsilon = (\epsilon_0, \epsilon_1, \dots)$ and $(\delta_0, \delta_1, \dots)$ are arbitrary sequence of constant parameters. Since, obviously, all the elements P_n are linearly independent, we conclude that each Hamiltonian PDE $\frac{du}{dt} = P_n$, $n \in \mathbb{Z}_+$, is integrable, associated to an integrable Lenard-Magri scheme of C-type.

Similarly, in the case $a_2 = 0, a_3 \neq 0$, all the sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$, $\{P_n\}_{n \in \mathbb{Z}_+}$, satisfying the Lenard-Magri recursive relations (2.18), are as follows:

$$(3.31) \qquad P_n = p_n(u; a_1, a_2, \epsilon) u' + \delta_n u'\,, \quad \int h_n = \int \left(\frac{d}{du} \right)^{-2} p_n(u; a_1, a_2, \epsilon)\,,$$

where $\epsilon = (\epsilon_0, \epsilon_1, \dots)$ and $(\delta_0, \delta_1, \dots)$ are arbitrary sequences of constant parameters. Again, we conclude that each Hamiltonian PDE $\frac{du}{dt} = P_n$, $n \in \mathbb{Z}_+$, is integrable, associated to an integrable Lenard-Magri scheme of C-type.

3.6 Summary

Let us summarize the results from the previous sections by listing all the possibilities for the pairs H and K as in (3.1), and specifying, using the terminology of Section 2.4, whether the corresponding Lenard-Magri sequence

$$(3.32) \qquad \textstyle\int 0 \xleftrightarrow{H} P_0 \xleftrightarrow{K} \int h_0 \xleftrightarrow{H} P_1 \xleftrightarrow{K} \int h_1 \xleftrightarrow{H} P_2 \xleftrightarrow{K} \dots ,$$

is *integrable of S-type*, i.e. the $\mathcal{C}$-span of the elements P_n's and $\int h_n$'s is infinite dimensional and H has order strictly greater than K ($b_1 = 0, a_1 \neq 0$), whether it is *integrable of C_1-type*, i.e. the $\mathcal{C}$-span of the elements P_n's and $\int h_n$'s is infinite dimensional and the orders of H and K are both equal to -1 ($b_1 = a_1 = 0$), whether it is *integrable of C_2-type*, i.e. the $\mathcal{C}$-span of the elements P_n's and $\int h_n$'s is infinite dimensional and H has order less than or equal to K and K has order 1 ($b_1 \neq 0$), whether it is of *finite type*, i.e. the $\mathcal{C}$-span of the elements P_n's or $\int h_n$'s is necessarily finite dimensional, or whether it is *blocked*, i.e. there are choices of $\int h_n$ or P_n for which the scheme cannot be continued. This is the list of all possibilities:

- *integrable of S-type*:

  (a) $b_1 = 0$, $(b_2, b_3) \neq (0, 0)$, $a_1 a_2 a_3 \neq 0$;
  (b) $b_1 = 0$, $a_1 \neq 0$, and either $b_2 \neq 0$, $a_2 = 0$ and $(b_3, a_3) \neq (0, 0)$, or $b_3 \neq 0$, $a_3 = 0$ and $(b_2, a_2) \neq (0, 0)$.

- *integrable of C_1-type*:

  $b_1 = a_1 = 0$, and either $b_2 a_3 \neq 0$ and (b_3, a_2) arbitrary, or $b_3 a_2 \neq 0$ and (b_2, a_3) arbitrary.

- *integrable of C_2-type*:

  (a) $b_1 a_1 \neq 0$, and either $b_2 = a_2 = 0$ and $(b_3, a_3) \neq (0, 0)$, or $b_3 = a_3 = 0$ and $(b_2, a_2) \neq (0, 0)$;
  (b) $b_1 \neq 0$, $a_1 = 0$, and either $b_2 = a_2 = 0$ and $a_3 \neq 0$ (with b_3 arbitrary), or $b_3 = a_3 = 0$ and $a_2 \neq 0$ (with b_2 arbitrary).

- *finite type*:

  (a) $b_1 b_2 b_3 \neq 0$, $a_1 = 0$, $(a_2, a_3) \neq (0, 0)$;
  (b) $b_1 = 0$, $a_1 \neq 0$, and either $b_2 = a_2 = 0$ and $b_3 \neq 0$ (with a_3 arbitrary), or $b_3 = a_3 = 0$ and $b_2 \neq 0$ (with a_2 arbitrary).

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(c1) $b_1 = a_1 = 0$, and either $b_2 = a_2 = 0$ and $b_3 a_3 \neq 0$, or $b_2 a_2 \neq 0$ and $b_3 = a_3 = 0$;

(c2) $b_1 b_2 b_3 \neq 0, a_1 a_2 a_3 \neq 0$;

(d) $b_1 b_2 b_3 \neq 0, a_1 \neq 0, a_2 a_3 = 0$.

- *blocked*:

  (a) $b_1 \neq 0$, $a_1 = 0$, and either $b_2 = 0$, $a_2 \neq 0$ and $(b_3, a_3) \neq (0,0)$, or $b_3 = 0$, $a_3 \neq 0$ and $(b_2, a_2) \neq (0,0)$;

  (b) $b_1 \neq 0, b_2 b_3 = 0, a_1 a_2 a_3 \neq 0$;

  (c) $b_1 a_1 \neq 0$, and either $b_2 a_3 \neq 0$ and $b_3 = a_2 = 0$, or $b_2 = a_3 = 0$ and $b_3 a_2 \neq 0$.

3.7 Going to the left

Suppose we have an integrable Lenard-Magri sequence (3.32) (of S, C_1 or C_2-type). A natural question is whether this sequence can be continued to the left:

(3.33)
$$\ldots \xleftrightarrow{H} P_{-1} \xleftrightarrow{K} \int 0 \xleftrightarrow{H} P_0 \xleftrightarrow{K} \int h_0 \xleftrightarrow{H} P_1 \xleftrightarrow{K} \int h_1 \xleftrightarrow{H} P_2 \xleftrightarrow{K} \ldots .$$

In this way we get some additional equations compatible with the given hierarchy $\frac{du}{dt_n} = P_n$, $n \in \mathbb{Z}_+$, and additional integrals of motion $\int h_n$, $n = -1, -2, \ldots$, in involution with the given $\int h_n$'s, with $n \geq 0$.

Clearly, trying to extend the Lenard-Magri scheme (3.33) to the left amounts to switching the roles of the non-local Poisson structures H and K, and to constructing the "dual" Lenard-Magri sequence

(3.34) $\quad \int 0 \xleftrightarrow{K} P_{-1} \xleftrightarrow{H} \int h_{-1} \xleftrightarrow{K} P_{-2} \xleftrightarrow{H} \int h_{-2} \xleftrightarrow{K} P_{-3} \xleftrightarrow{H} \ldots .$

So, we need to study, for each possible choice of the parameters $a_i, b_i, i = 1, 2, 3$, what type of Lenard-Magri scheme we get when we switch all the coefficients a_i's with the b_i's.

By looking at the list of all possibilities in the previous section, after switching the roles of H and K we have the following:

- integrable of S-type (a) $\xleftrightarrow{H \leftrightarrow K}$ finite type (a);

- integrable of S-type (b) $\xleftrightarrow{H \leftrightarrow K}$ blocked (a);

- integrable of C_1-type $\xleftrightarrow{H \leftrightarrow K}$ integrable of C_1-type;

- integrable of C_2-type (a) $\overset{H\leftrightarrow K}{\longleftrightarrow}$ integrable of C_2-type (a);

- integrable of C_2-type (b) $\overset{H\leftrightarrow K}{\longleftrightarrow}$ finite-type (b);

- finite-type (c) $\overset{H\leftrightarrow K}{\longleftrightarrow}$ finite-type (c);

- finite-type (d) $\overset{H\leftrightarrow K}{\longleftrightarrow}$ blocked (b);

- blocked (c) $\overset{H\leftrightarrow K}{\longleftrightarrow}$ blocked (c).

We are only interested in the integrable (S or C-type) Lenard-Magri schemes. We see from the above list that, after exchanging the roles of H and K, three things can happen. The "dual" Lenard-Magri scheme (3.34) can be of *finite*-type (this happens in the cases S(a) and C_2(b)). In this situation continuing the Lenard-Magri scheme to the left we never get any new interesting integrals of motion or equations.

The second possibility is that the "dual" Lenard-Magri scheme (3.34) is of *integrable*-type (this happens in the cases C_1 and C_2(a)). In this situation we can continue the Lenard-Magri scheme to the left indefinitely. In other words, in each of these cases we can merge two integrable systems, "dual" to each other, to get one integrable system with twice as many integrals of motion and equations.

The most interesting situation is when the "dual" Lenard-Magri scheme (3.34) is *blocked* (which happens in the case S(b)). In this case, if the sequence (3.34) is blocked at P_{-k}, $k \geq 0$, we obtain an integrable PDE which is not of evolutionary type.

3.8 Non-evolutionary integrable equations

According to the previous discussion, we need to consider the case of integrable Lenard-Magri scheme of S-type (b), which means the following 5 cases:

1. $b_1 = 0, b_2 \neq 0, b_3 \neq 0, a_1 \neq 0, a_2 \neq 0, a_3 = 0$;

2. $b_1 = 0, b_2 \neq 0, b_3 \neq 0, a_1 \neq 0, a_2 = 0, a_3 \neq 0$;

3. $b_1 = 0, b_2 \neq 0, b_3 \neq 0, a_1 \neq 0, a_2 = 0, a_3 = 0$;

4. $b_1 = 0, b_2 \neq 0, b_3 = 0, a_1 \neq 0, a_2 = 0, a_3 \neq 0$;

5. $b_1 = 0, b_2 = 0, b_3 \neq 0, a_1 \neq 0, a_2 \neq 0, a_3 = 0$;

Case 1: $b_1 = 0, b_2 \neq 0, b_3 \neq 0, a_1 \neq 0, a_2 \neq 0, a_3 = 0$

This case gives, to the right, the Lenard-Magri scheme listed as case (ii) in Section 3.3, while, after exchanging the roles of H and K we get, to the left, the "blocked" Lenard-Magri scheme listed as case 3 in Section 3.5. Hence, overall, we get the following scheme:

$$\nexists P \xleftrightarrow{K} \int e^{\pm a_{12} x} u \xleftrightarrow{H} 0 \xleftrightarrow{K} \int 0 \xleftrightarrow{H} 1 \xleftrightarrow{K} \int \sqrt{b_2 + b_3 (u')^2} \xleftrightarrow{H}$$

$$\xleftrightarrow{H} -\frac{a_1 b_2 b_3 u'''}{(b_2 + b_3 (u')^2)^{\frac{3}{2}}} + 3 \frac{a_1 b_2 b_3^2 u'(u'')^2}{(b_2 + b_3 (u')^2)^{\frac{5}{2}}} - \frac{a_2 b_3}{\sqrt{b_2 + b_3 (u')^2}} \xleftrightarrow{K} \dots .$$

Here and further $\pm$ means that we take arbitrary linear combination of the above expressions with $+$ and with $-$. Trying to solve naively for P in the above scheme, we get the following expression

$$P = \pm \frac{b_2}{a_{12}} e^{\pm a_{12} x} + b_3 u' \partial^{-1} \left(e^{\pm a_{12} x} u' \right).$$

The meaning of the above expression for P is that the following partial differential equation is a member of the integrable hierarchy associated to the Lenard-Magri scheme of S-type (ii):

$$(3.35) \qquad \left(\frac{u_t}{u_x} \right)_x = \pm \frac{b_2}{a_{12}} \left(\frac{1}{u_x} e^{\pm a_{12} x} \right)_x + b_3 e^{\pm a_{12} x} u_x .$$

Case 2: $b_1 = 0, b_2 \neq 0, b_3 \neq 0, a_1 \neq 0, a_2 = 0, a_3 \neq 0$

This case gives, to the right, the Lenard-Magri scheme listed as case (iii) in Section 3.3, while, after exchanging the roles of H and K we get, to the left, the "blocked" Lenard-Magri scheme listed as case 5 in Section 3.5. Hence, overall, we get the following scheme:

$$\nexists P \xleftrightarrow{K} \int e^{\pm a_{13} u} \xleftrightarrow{H} 0 \xleftrightarrow{K} \int 0 \xleftrightarrow{H} u' \xleftrightarrow{K} \int \sqrt{b_2 + b_3 (u')^2} \xleftrightarrow{H}$$

$$\xleftrightarrow{H} -\frac{a_1 b_2 b_3 u'''}{(b_2 + b_3 (u')^2)^{\frac{3}{2}}} + 3 \frac{a_1 b_2 b_3^2 u'(u'')^2}{(b_2 + b_3 (u')^2)^{\frac{5}{2}}} + \frac{a_3 b_2}{\sqrt{b_2 + b_3 (u')^2}} \xleftrightarrow{K} \dots .$$

Trying to solve naively for P we get

$$P = \pm a_{13} b_2 \partial^{-1} e^{\pm a_{13} u} + b_3 u' e^{\pm a_{13} u} .$$

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This means that the following hyperbolic partial differential equation is a member of the integrable hierarchy associated to the Lenard-Magri scheme of S-type (iii):

$$(3.36) \qquad u_{tx} = \pm a_{13} b_2 e^{\pm a_{13} u} \pm \frac{b_3}{a_{13}} \left(e^{\pm a_{13} u} \right)_{xx}.$$

Case 3: $b_1 = 0, b_2 \neq 0, b_3 \neq 0, a_1 \neq 0, a_2 = 0, a_3 = 0$

This case gives, to the right, the Lenard-Magri scheme listed as case (iv) in Section 3.3, while, after exchanging the roles of H and K we get, to the left, the "blocked" Lenard-Magri scheme listed as case 6 in Section 3.5. Hence, overall, we get, depending on how we choose to continue the scheme to the left, the following two possibilities (or any their linear combination):

$$\nexists P \xleftrightarrow{K} \frac{1}{a_1} \int xu \xleftrightarrow{H} 1 \xleftrightarrow{K} \int 0 \xleftrightarrow{H} 0 \xleftrightarrow{K} \int \sqrt{b_2 + b_3 (u')^2} \xleftrightarrow{H}$$

$$\xleftrightarrow{H} -\frac{a_1 b_2 b_3 u'''}{(b_2 + b_3 (u')^2)^{\frac{3}{2}}} + 3 \frac{a_1 b_2 b_3^2 u' (u'')^2}{(b_2 + b_3 (u')^2)^{\frac{5}{2}}} \xleftrightarrow{K} \dots ,$$

or

$$\nexists P \xleftrightarrow{K} \frac{1}{a_1} \int \frac{1}{2} u^2 \xleftrightarrow{H} u' \xleftrightarrow{K} \int 0 \xleftrightarrow{H} 0 \xleftrightarrow{K} \int \sqrt{b_2 + b_3 (u')^2} \xleftrightarrow{H}$$

$$\xleftrightarrow{H} -\frac{a_1 b_2 b_3 u'''}{(b_2 + b_3 (u')^2)^{\frac{3}{2}}} + 3 \frac{a_1 b_2 b_3^2 u' (u'')^2}{(b_2 + b_3 (u')^2)^{\frac{5}{2}}} \xleftrightarrow{K} \dots .$$

Trying to solve naively for P we get, in the first case

$$P = \frac{b_2}{2a_1} x^2 + \frac{b_3}{a1} xuu' - \frac{b_3}{a_1} u' \partial^{-1} u,$$

which corresponds to the following integrable non-evolutionary partial differential equation:

$$(3.37) \qquad \left(\frac{u_t}{u_x} \right)_x = \frac{b_2}{2a_1} \left(\frac{x^2}{u_x} \right)_x + \frac{b_3}{a_1} xu_x.$$

In the second case we get

$$P = \frac{b_2}{a_1} \partial^{-1} u + \frac{b_3}{2a_1} u^2 u',$$

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which corresponds to the following integrable hyperbolic partial differential equation:

$$(3.38) \qquad u_{tx} = \frac{b_2}{a_1} u + \frac{b_3}{6a_1} (u^3)_{xx} \, .$$

In conclusion, both equations (3.37) and (3.38) are members of the integrable hierarchy associated to the Lenard-Magri scheme of S-type (iv).

Case 4: $b_1 = 0, b_2 \neq 0, b_3 = 0, a_1 \neq 0, a_2 = 0, a_3 \neq 0$

This case gives, to the right, the Lenard-Magri scheme listed as case (vii) in Section 3.3, while, after exchanging the roles of H and K we get, to the left, the "blocked" Lenard-Magri scheme listed as case 5 in Section 3.5. Hence, overall, we get the following scheme:

$$\nexists P \xleftrightarrow{K} \int e^{\pm a_{13} u} \xleftrightarrow{H} 0 \xleftrightarrow{K} \int 0 \xleftrightarrow{H} u' \xleftrightarrow{K} \int \frac{-(u')^2}{2b_2} \xleftrightarrow{H}$$

$$\xleftrightarrow{H} \frac{a_1}{b_2} u''' + \frac{a_3}{2b_2} (u')^3 \xleftrightarrow{K} \ldots \, .$$

Trying to solve naively for P we get

$$P = \pm a_{13} b_2 \partial^{-1} e^{\pm a_{13} u} \, .$$

This means that the following hyperbolic partial differential equation is a member of the integrable hierarchy associated to the Lenard-magri scheme of S-type (vii):

$$(3.39) \qquad u_{tx} = \pm a_{13} b_2 e^{\pm a_{13} u} \, .$$

As expected, equation (3.39) is obtained by (3.36) letting $b_3 = 0$.

Case 5: $b_1 = 0, b_2 = 0, b_3 \neq 0, a_1 \neq 0, a_2 \neq 0, a_3 = 0$

This case gives, to the right, the Lenard-Magri scheme listed as case (x) in Section 3.3, while, after exchanging the roles of H and K we get, to the left, the "blocked" Lenard-Magri scheme listed as case 3 in Section 3.5. Hence, overall, we get the following scheme:

$$\nexists P \xleftrightarrow{K} \int e^{\pm a_{12} x} u \xleftrightarrow{H} 0 \xleftrightarrow{K} \int 0 \xleftrightarrow{H} 1 \xleftrightarrow{K} \int \frac{1}{2b_3 u'} \xleftrightarrow{H}$$

$$\xleftrightarrow{H} -\frac{a_1}{b_3} \frac{u'''}{(u')^3} + \frac{3a_1}{b_3} \frac{(u'')^2}{(u')^4} + \frac{a_2}{2b_3} \frac{1}{(u')^2} \xleftrightarrow{K} \ldots \, .$$

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Trying to solve naively for P in the above scheme, we get the following expression

$$P = b_3 u' \partial^{-1} \left(e^{\pm a_{12}x} u' \right),$$

and the associated non-evolutionary partial differential equation is

(3.40)
$$\left(\frac{u_t}{u_x} \right)_x = b_3 e^{\pm a_{12}x} u_x .$$

In conclusion, equation (3.40) is a member of the integrable hierarchy associated to the Lenard-Magri scheme of S-type (x). Note that this equation is obtained letting $b_2 = 0$ in equation (3.35).

Conclusion

After rescaling the variables u, x and t, or replacing x by $x+$ const., or u by $u+$ const., in equations (3.35)-(3.40), we conclude that the following are all the integrable non-evolutionary partial differential equations which are members of some integrable hierarchy of bi-Hamiltonian equations, with H and K as in (3.1):

(3.41)
$$u_{tx} = e^u - \alpha e^{-u} + \epsilon(e^u - \alpha e^{-u})_{xx} ,$$

(3.42)
$$\left(\frac{u_t}{u_x} \right)_x = (e^x - \alpha e^{-x})u_x + \epsilon\left(\frac{e^x - \alpha e^{-x}}{u_x} \right)_x ,$$

(3.43)
$$u_{tx} = u + (u^3)_{xx} ,$$

(3.44)
$$\left(\frac{u_t}{u_x} \right)_x = \left(\frac{x^2}{u_x} \right)_x + xu_x ,$$

where α and ϵ are 0 or 1.

Recall that the case when $\epsilon = 0$ equation (3.41) is the Liouville equation when $\alpha = 0$, and the sinh-Gordon equation when $\alpha = 1$, cf. [Dor93]. Equation (3.42) (respectively (3.44)) can be obtained from equation (3.41) (resp. (3.43)) by the hodograph transformation $u \mapsto x$, $x \mapsto -u$. Equation (3.43) is called the "short pulse equation", [SW02], and its integrability was proved in [SS04]. Equations (3.41) with $\epsilon = 1$ seems to be new.

4 KN type integrable system

In this section $\mathcal{V}$ is a field of differential functions in u, and, as usual, we assume that $\mathcal{V}$ contains all the functions that we encounter in our computations.

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Recall from [DSK12, Example 4.6] that the following is a pair of compatible non-local Poisson structures:

$$L_1 = u'\partial^{-1} \circ u' \ \ (\text{Sokolov}) \ \ , \ \ L_2 = \partial^{-1} \circ u'\partial^{-1} \circ u'\partial^{-1} \ \ (\text{Dorfman}) \ .$$

We consider two non-local Poisson structures H and K which are linear combinations of L_1 and L_2: $H = a_1 L_1 + a_2 L_2$ and $K = b_1 L_1 + b_2 L_2$. As we have seen in the example of Liouville type integrable systems, discussed in Section 3, integrable hierarchies associated to Lenard-Magri schemes of C type are usually not very interesting (cf. Sections 3.4 and 3.5). Hence, in this section, we will only consider integrable Lenard-Magri schemes of S-type (in the terminology of Section 2.4), which is possible only when the order of the pseudodifferential operator H is greater that the order of K, namely when $a_1 \neq 0$ and $b_1 = 0$. Therefore, we consider the following compatible pair of non-local structures:

$$(4.1) \quad H = u'\partial^{-1} \circ u' + a\partial^{-1} \circ u'\partial^{-1} \circ u'\partial^{-1} \ , \ \ K = \partial^{-1} \circ u'\partial^{-1} \circ u'\partial^{-1} \ ,$$

with $a \in \mathcal{C}$. We want to discuss the integrability of the corresponding Lenard-Magri scheme.

4.1 Preliminary computations

Note that K is the inverse of a differential operator, hence its minimal fractional decomposition is $K = 1D^{-1}$, where

$$(4.2) \qquad\qquad D = \partial \circ \frac{1}{u'}\partial \circ \frac{1}{u'}\partial \ .$$

We next find a minimal fractional decomposition for H. It is given by the following

Lemma 4.1. *For every $a \in \mathcal{C}$, we have $H = AB^{-1}$, where*

$$(4.3) \qquad
\begin{aligned}
A &= \left(\partial^2 - 2\frac{u''}{u'}\partial + \left(\frac{u''}{u'}\right)' + a \right) \circ \frac{1}{D(u')}\partial - u' \ , \\
B &= \partial \circ \frac{1}{u'}\partial \circ \frac{1}{u'}\partial \circ \frac{1}{D(u')}\partial \ .
\end{aligned}$$

Here and further, we have, recalling (4.2),

$$(4.4) \qquad\qquad D(u') = \left(\frac{1}{u'} \left(\frac{u''}{u'} \right)' \right)' \ .$$

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The above fractional decomposition is minimal only for $a \neq 0$. For $a = 0$, the minimal fractional decomposition for H is $H = 1S^{-1}$, where

$$(4.5) \qquad S = \frac{1}{u'}\partial \circ \frac{1}{u'}.$$

Proof. We need to prove that $AB^{-1} = S^{-1} + aD^{-1}$. By looking at the coefficient of a in AB^{-1}, we get

$$\frac{1}{D(u')}\partial\left(\partial \circ \frac{1}{u'}\partial \circ \frac{1}{u'}\partial \circ \frac{1}{D(u')}\partial\right)^{-1} = \partial^{-1} \circ u'\partial^{-1} \circ u'\partial^{-1} = D^{-1}.$$

Letting $a = 0$ in AB^{-1}, we have

$$\left(\left(\partial^2 - 2\frac{u''}{u'}\partial + \left(\frac{u''}{u'}\right)'\right) \circ \frac{1}{D(u')}\partial - u'\right)\left(\partial \circ \frac{1}{u'}\partial \circ \frac{1}{u'}\partial \circ \frac{1}{D(u')}\partial\right)^{-1}$$

$$= \left(\partial^2 - 2\frac{u''}{u'}\partial + \left(\frac{u''}{u'}\right)' - u'\partial^{-1} \circ D(u')\right) \circ \partial^{-1}u'\partial^{-1} \circ u'\partial^{-1}$$

$$= \partial \circ u'\partial^{-1} \circ u'\partial^{-1} - 2u''\partial^{-1} \circ u'\partial^{-1} + \left(\frac{u''}{u'}\right)'\partial^{-1}u'\partial^{-1} \circ u'\partial^{-1}$$

$$-u'\partial^{-1} \circ D(u')\partial^{-1}u'\partial^{-1} \circ u'\partial^{-1} = u'\partial^{-1} \circ u' + \left(u'\partial^{-1} \circ \frac{u''}{u'}\right.$$

$$\left. -u''\partial^{-1} + \left(\frac{u''}{u'}\right)'\partial^{-1}u'\partial^{-1} - u'\partial^{-1} \circ D(u')\partial^{-1}u'\partial^{-1}\right) \circ u'\partial^{-1}.$$

In the last identity we used the Leibniz rule for ∂: $\partial \circ f = f\partial + f'$. To conclude the proof, we need to check that the expression in parenthesis in the RHS is zero:

$$(4.6) \quad u'\partial^{-1} \circ \frac{u''}{u'} - u''\partial^{-1} + \left(\frac{u''}{u'}\right)'\partial^{-1}u'\partial^{-1} - u'\partial^{-1} \circ D(u')\partial^{-1}u'\partial^{-1} = 0.$$

This identity is obtained applying repeatedly the commutation relation ($f \in \mathcal{V}$),

$$(4.7) \qquad \partial^{-1} \circ f = f\partial^{-1} - \partial^{-1} \circ f'\partial^{-1},$$

which is a consequence of the Leibniz rule for ∂, and using the expression (4.4) for $D(u')$. $\qquad\square$

In order to apply successfully the Lenard-Magri scheme of integrability we need to compute the kernel of the operator B.

Lemma 4.2. *The kernel of the operator B in (4.3) is a 4-dimensional vector space over $\mathcal{C}$, spanned by*

$$f_1 = 1 \,, \quad f_2 = \frac{1}{u'}\left(\frac{u''}{u'}\right)' \,, \quad f_3 = \frac{u}{u'}\left(\frac{u''}{u'}\right)' - \frac{u''}{u'} \,,$$
$$f_4 = \frac{u^2}{u'}\left(\frac{u''}{u'}\right)' - 2u\frac{u''}{u'} + 2u' \,.$$

Proof. It is immediate to check that all the elements f_i are indeed in the kernel of B. On the other hand, since B has order 4, its kernel has dimension at most 4. $\square$

4.2 Applying the Lemard-Magri scheme for $a \neq 0$

According to the Lenard-Magri scheme of integrability, starting with $\int h_{-1} = \int 0$, we need to find sequences $\{\int h_n\}_{n=0}^N$ and $\{P_n\}_{n=0}^N$ solving the recursion conditions (2.16).

Since $C = 1$, in order to find solutions of the scheme (2.16) for $N = 3$, we need to find elements $F_n, h_n, P_n \in \mathcal{V}$, $n = 0, \ldots, 3$, such that

$$BF_n = \frac{\delta h_{n-1}}{\delta u} \,, \quad P_n = AF_n \,, \quad \frac{\delta h_n}{\delta u} = DP_n \,,$$

for all $n = 0, 1, 2, 3$ (we let, as usual, $\int h_{-1} = \int 0$). Recalling the expressions (4.2) and (4.3) of A, B, D, and using Lemma 4.2, it is straightforward but lengthy calculation to find solutions:

$$F_0 = \frac{f_2}{a} = \frac{1}{au'}\left(\frac{u''}{u'}\right)' \,, \qquad\qquad P_0 = 1 \,, \quad \int h_0 = \int 0 \,,$$
$$F_1 = \frac{f_3}{a} = \frac{u}{au'}\left(\frac{u''}{u'}\right)' - \frac{u''}{u'} \,, \qquad P_1 = u \,, \quad \int h_1 = \int 0 \,,$$
$$F_2 = \frac{f_4}{a} = \frac{u^2}{au'}\left(\frac{u''}{u'}\right)' - 2\frac{uu''}{au'} + \frac{2}{a}u' \,, \quad P_2 = u^2 \,, \quad \int h_2 = \int 0 \,,$$
$$F_3 = -f_1 = -1 \,, \qquad\qquad P_3 = u' \,, \quad \int h_3 = \frac{1}{2}\int\left(\frac{u''}{u'}\right)^2 \,.$$

Hence, we get the following Lenard-Magri scheme

(4.8)
$$\int 0 \xleftrightarrow{H} 1 \xleftrightarrow{K} \int 0 \xleftrightarrow{H} u \xleftrightarrow{K} \int 0 \xleftrightarrow{H} u^2 \xleftrightarrow{K} \int 0 \xleftrightarrow{H}$$
$$\xleftrightarrow{H} u' \xleftrightarrow{K} \frac{1}{2}\int\left(\frac{u''}{u'}\right)^2 \xleftrightarrow{K} \ldots \,.$$

We next prove that the scheme (4.8) can be extended indefinitely. According to Theorem 2.11, this is the case, provided that the orthogonality

conditions (2.17) hold. Since $C = 1$, the first condition in (2.17) is trivial. As for the second orthogonality condition, let $\varphi \in \left(\mathrm{Span}_C \{P_0, P_1, P_2, P_3\} \right)^{\perp}$. Since $\varphi \perp P_0$, we have that $\varphi = \varphi_1'$, for some $\varphi_1 \in \mathcal{V}$. Since $\varphi \perp P_1$, we have that $\varphi_1 = \frac{\varphi_2'}{u'}$, for some $\varphi_2 \in \mathcal{V}$. Since $\varphi \perp P_2$, we have that $\varphi_2 = \frac{\varphi_3'}{u'}$, for some $\varphi_3 \in \mathcal{V}$. And, finally, since $\varphi \perp P_3$, we have that $\varphi_3 = \frac{\varphi_4'}{D(u')}$, for some $\varphi_4 \in \mathcal{V}$. In conclusion, $\varphi = B\varphi_4$, proving the second orthogonality condition (2.17).

We compute explicitly the next element P_4 in the Lenard scheme, which gives the first non-trivial equation of the corresponding bi-Hamiltonian hierarchy. For this, we need to solve, for $F_4, P_4 \in \mathcal{V}$, the following equations

$$BF_4 = \frac{\delta h_3}{\delta u} = D(u') \,, \quad P_4 = AF_4 \,.$$

The general solution is:

$$F_4 = \left(\frac{u''}{u'}\right)' - \frac{1}{2}\left(\frac{u''}{u'}\right)^2 + (a - \alpha_1)f_1 + \frac{\alpha_2}{a}f_2 + \frac{\alpha_3}{a}f_3 + \frac{\alpha_4}{a}f_4 \,,$$

where α_i, $i = 1, \ldots, 4$, are arbitrary constants. Hence, the first non-trivial integrable equation in the hierarchy has the form:

$$(4.9) \qquad \frac{du}{dt} = P_4 = u''' - \frac{3}{2}\frac{(u'')^2}{u'} + \alpha_1 u' + \alpha_2 + \alpha_3 u + \alpha_4 u^2 \,.$$

In order to prove that equation (4.9) is indeed integrable, we are left to prove that the sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$ and $\{P_n\}_{n \in \mathbb{Z}_+}$ are linearly independent. For this, we use Proposition 2.15.

Since $\mathrm{dord}(D(u')) = 4$, we have $\mathrm{dord}(A) = 6$, $\mathrm{dord}(B) = 7$, $\mathrm{dord}(C) = -\infty$ and $\mathrm{dord}(D) = 3$. Moreover, we clearly have $|H| = -1$ and $|K| = -3$. Hence, the RHS of inequality (2.23) is 4.

Next, we compute the differential order of the next element P_5 in the Lenard-Magri scheme. It is obtained by solving, for $\xi_4 = \frac{\delta h_4}{\delta u}, F_5, P_5 \in \mathcal{V}$, the following equations:

$$(4.10) \qquad \xi_4 = DP_4 \,, \quad BF_5 = \xi_4 \,, \quad P_5 = AF_5 \,.$$

From the first equation in (4.10) we get

$$\xi_4 = \partial \frac{1}{u'} \partial \frac{1}{u'} \partial (u''' + \rho) \,,$$

where $\rho \in \mathcal{V}$ has $\mathrm{dord}(\rho) = 2$. Hence, the second equation in (4.10) gives

$$\frac{1}{D(u')} \partial F_5 = u''' + \rho_1 \,,$$

where dord$(\rho_1 - \rho) = 0$. In particular, F_5 has differential order less than or equal to 3. It follows by the third equation in (4.10) that

$$P_5 = \left(\partial^2 - 2\frac{u''}{u'}\partial + \left(\frac{u''}{u'}\right)' + a \right)(u''' + \rho_1) - u'F_5 \,.$$

Hence, $\frac{\partial P_5}{\partial u^{(5)}} = 1$, and $\frac{\partial P_5}{\partial u^{(n)}} = 0$ for every $n > 5$. In particular, dord$(P_5) = 5$.

According to Proposition 2.15, since dord$(P_5) = 5 > 4$, we have that dord$\left(\frac{\delta h_n}{\delta u}\right) = 2n - 2$, and dord$(P_n) = 2n - 5$, for every $n \geq 3$. In particular, all the elements $\{\int h_n\}_{n\in\mathbb{Z}_+}$ and $\{P_n\}_{n\in\mathbb{Z}_+}$ are linearly independent. As a consequence, every equation of the hierarchy $\frac{du}{dt_n} = P_n$, $n \in \mathbb{Z}_+$, including equation (4.9), is integrable.

Note that, since the kernels of B^* and D^* have non-zero intersections, we cannot conclude that $[P_m, P_n]$ is zero for every $m, n \in \mathbb{Z}_+$ (cf. Theorem 2.11), but we believe this is true.

When all constants α_i are equal to zero, equation (4.9) is called in [Dor93] the Krichever-Novikov (KN) equation. As explained in [MS12], equation (4.9) can be reduced to the KN equation by some point transformation.

4.3 The case $a = 0$

In the case when $a = 0$ all the computations are much easier. Since $A = C = 1$, the recursive conditions $\int h_{n-1} \xleftrightarrow{H} P_n \xleftrightarrow{K} \int h_n$, $n \in \mathbb{Z}_+$, are equivalent to the equations

$$BP_n = \frac{\delta h_{n-1}}{\delta u} \,, \quad \frac{\delta h_n}{\delta u} = DP_n \,.$$

It is easy to find the first few steps of the Lenard-Magri scheme:
(4.11)

$$\int 0 \xleftrightarrow{H} P_0 = u' \xleftrightarrow{K} \int h_0 = \frac{1}{2}\int \left(\frac{u''}{u'}\right)^2 \xleftrightarrow{H} P_1 = u''' - \frac{3}{2}\frac{(u'')^2}{u'} + \alpha_1 u' \xleftrightarrow{K} \dots$$

for arbitrary $\alpha_1 \in \mathcal{C}$.

As before, the scheme (4.11) can be extended indefinitely. Indeed, since $C = 1$, the first orthogonality condition in (2.17) is trivial, while the second one holds since $P_0^\perp = \mathrm{Im}\, B$.

Moreover, in this case dord$(A) = $ dord$(C) = -\infty$, dord$(B) = 2$, and dord$(D) = 3$, so the RHS of inequality (2.23) is 0. Since dord$(P_0) = 1 > 0$, we can apply Proposition 2.15 to deduce that all the elements $\{\int h_n\}_{n\in\mathbb{Z}_+}$ and $\{P_n\}_{n\in\mathbb{Z}_+}$ are linearly independent.

In conclusion, every equation of the hierarchy $\frac{du}{dt_n} = P_n$, $n \in \mathbb{Z}_+$, is integrable. Note that the first non-trivial equation is $\frac{du}{dt} = P_1$, which is the same as equation (4.9) with $\alpha_2 = \alpha_3 = \alpha_4 = 0$.

4.4 One step back

As we did in the example of Liouville type integrable systems, we can ask whether the Lenard-Magri schemes (4.8) and (4.11) can be continued to the left.

This amounts to finding $P_n \in \mathcal{V}$ and $\int h_n \in \mathcal{V}/\partial\mathcal{V}$, with $n \leq -1$, such that

$$(4.12) \qquad \dots \xleftrightarrow{K} \int h_{-2} \xleftrightarrow{H} P_{-1} \xleftrightarrow{K} \int 0$$

We consider separately the cases $a \neq 0$ and $a = 0$. When $a \neq 0$, the conditions (4.12) give the following equations for P_{-1}, F and $\xi_{-2} = \frac{\delta h_{-2}}{\delta u}$:

$$(4.13) \qquad DP_{-1} = 0 \ , \quad AF = P_{-1} \ , \quad \xi_{-2} = BF \ ,$$

where A, B, and D are as in (4.2) and (4.3). All solutions P_{-1} of the first equation in (4.13) are

$$P_{-1} = c_0 + c_1 u + c_2 u^2 \ ,$$

with $c_0, c_1, c_2 \in \mathcal{C}$. Next, we want to find all solutions F of the second equation in (4.13). Applying $\frac{\partial}{\partial u^{(n)}}$, with $n \geq 4$, to both sides of the equation $AF = P_{-1}$ we immediately get that $\mathrm{dord}(F) \leq 3$ and $\partial F = fD(u')$, with $\mathrm{dord}(f) \leq 1$. Hence, the second equation in (4.13) can be rewritten as the following system of equations,

$$(4.14) \qquad \begin{aligned} &\partial^2 f - 2\frac{u''}{u'}\partial f + \left(\frac{u''}{u'}\right)'f + af - u'F = c_0 + c_1 u + c_2 u^2 \ , \\ &\partial F = fD(u') \ , \end{aligned}$$

for $F, f \in \mathcal{V}$ with $\mathrm{dord}(F) \leq 3$ and $\mathrm{dord}(f) \leq 1$. Applying $\frac{\partial}{\partial u^{(3)}}$ to both sides of the first equation in (4.14) and $\frac{\partial}{\partial u^{(4)}}$ to both sides of the second equation in (4.14), we get

$$\frac{\partial F}{\partial u'''} = \frac{f}{(u')^2} \ , \quad \frac{\partial f}{\partial u'} = 0 \ .$$

Hence, $\mathrm{dord}(f) \leq 0$. Next, applying $\frac{\partial}{\partial u^{(2)}}$ to the first equation in (4.14) and $\frac{\partial}{\partial u^{(3)}}$ to the second equation in (4.14), we get

$$\frac{\partial F}{\partial u''} = -2\frac{u''}{(u')^3}f - \frac{1}{(u')^2}\partial f \ , \quad \partial f = \frac{\partial f}{\partial u}u' \ .$$

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Hence, f is a function of u only. Using the above result, we can rewrite the second equation in (4.14), after integrating by parts twice, as

$$\partial F = \partial\Big(\frac{f}{u'}\Big(\frac{u''}{u'}\Big)' - \frac{\partial f}{\partial u}\frac{u''}{u'} + \frac{\partial^2 f}{\partial u^2}u'\Big) - \frac{\partial^3 f}{\partial u^3}(u')^2 \, .$$

In particular, it must be $\frac{\partial^3 f}{\partial u^3}(u')^2 \in \partial\mathcal{V}$, which is possible only if $\frac{\partial^3 f}{\partial u^3}(u')^2 = 0$, see [BDSK09]. In conclusion, f must be a quadratic polynomial in u with constant coefficients, and $F = \frac{f}{u'}\Big(\frac{u''}{u'}\Big)' - \frac{\partial f}{\partial u}\frac{u''}{u'} + \frac{\partial^2 f}{\partial u^2}u'+$ const. Plugging these results back into equation (4.14) we finally get that

$$\frac{\partial F}{D(u')} = f = \frac{c_0}{a} + \frac{c_1}{a}u + \frac{c_2}{a}u^2 \, .$$

Hence, the third equation in (4.13) gives $\xi_{-2} = 0$. In conclusion, in this case the "dual" Lenard-Magri sequence, obtained by exchanging the roles of H and K, is of *finite* type, namely it repeats itself with $\int h_n \in \mathrm{Ker}\left(\frac{\delta}{\delta u}\right)$ and $P_n \in \mathrm{Ker}(D)$ for every $n \leq -1$, and we don't get any new interesting integrals of motion or equations.

Next, we consider the case $a = 0$. In this case, the conditions (4.12) give the following equations for P_{-1}, P_{-2}, and $\int h_{-2}$:

$$(4.15) \qquad DP_{-1} = 0 \, , \quad DP_{-2} = \frac{\delta h_{-2}}{\delta u} = SP_{-1} \, ,$$

where S, and D are as in (4.2) and (4.5). As before, $P_{-1} = c_0 + c_1 u + c_2 u^2$, with $c_0, c_1, c_2 \in \mathcal{C}$. Hence, the second equation in (4.15) reads

$$(4.16) \qquad \Big(\frac{1}{u'}\Big(\frac{\partial P_{-2}}{u'}\Big)'\Big)' = \frac{1}{u'}\Big(\frac{c_0 + c_1 u + c_2 u^2}{u'}\Big)' \, .$$

For every $n \in \mathbb{Z}_+$, we have the identity

$$\frac{1}{u'}\Big(\frac{u^n}{u'}\Big)' = \frac{1}{2}\partial\frac{u^n}{(u')^2} + \frac{n}{2}\frac{u^{n-1}}{u'} \, .$$

It follows that the RHS of (4.16) cannot be a total derivative unless $c_1 = c_2 = 0$ (cf. [BDSK09]). Moreover, if $c_1 = c_2 = 0$ equation (4.16) reduces to

$$\Big(\frac{\partial P_{-2}}{u'}\Big)' = \frac{c_0}{2u'} + \text{ const.}u' \, ,$$

which, for the same reason as before, has no solutions unless $c_0 = 0$. In conclusion, for every non-zero $P_{-1} \in \mathrm{Ker}\, D$, the "dual" Lenard-Magri scheme is

blocked at P_{-2}. In this case, as we saw in Section 3.8, we obtain integrable PDE's which are not of evolutionary type.

In particular, for $(c_2, c_3) \neq (0,0)$ we get the following non-evolutionary integrable PDE:

$$(4.17) \qquad \left(\frac{1}{u_x} \left(\frac{u_{tx}}{u_x} \right)_x \right)_x = \frac{1}{u_x} \left(\frac{c_0 + c_1 u + c_2 u^2}{u_x} \right)_x ,$$

while for $c_1 = 1$ and $c_2 = c_3 = 0$, we obtain the following integrable equation

$$(4.18) \qquad \left(\frac{u_{tx}}{u_x} \right)_x = \frac{1}{2u_x} + \gamma u_x ,$$

where γ is a constant. Note that, if we apply the differential substitution $v = \log(u')$ to equation (3.41) with $\epsilon = 0$, we get equation (4.18).

5 NLS type integrable systems

Recall from [DSK12, Example 4.10] that the following is a triple of compatible non-local Poisson structures in two differential variables u, v:

$$L_1 = \partial \mathbb{1} , \quad L_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad L_3 = \begin{pmatrix} v\partial^{-1} \circ v & -v\partial^{-1} \circ u \\ -u\partial^{-1} \circ v & u\partial^{-1} \circ u \end{pmatrix} .$$

We want to consider two non-local Poisson structures H and K which are linear combinations of them: $H = a_1 L_1 + a_2 L_2 + a_3 L_3$ and $K = b_1 L_1 + b_2 L_2 + b_3 L_3$, where a_i's and b_i's are constants. As in the previous section, we are only interested in integrable Lenard-magri schemes of S-type. In particular, we assume that the order of the pseudodifferential operator H is greater that the order of K, and so we consider only the case when $b_1 = 0$. Note that when $b_2 = 0$, we get $K = b_3 L_3$, which is a degenerate pseudodifferential operator. In this case, we cannot apply Theorem 2.11 and we do not know how to prove integrability. Hence, we assume that $b_2 = 1$. And, since we want that the order of H is greater that the order of K, we assume also that $a_1 = 1$.

In conclusion, we consider the following compatible pair of non-local Hamiltonian structures:

$$(5.1) \qquad \begin{aligned} H &= \partial \mathbb{1} + a_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} v\partial^{-1} \circ v & -v\partial^{-1} \circ u \\ -u\partial^{-1} \circ v & u\partial^{-1} \circ u \end{pmatrix} \\ K &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + b_3 \begin{pmatrix} v\partial^{-1} \circ v & -v\partial^{-1} \circ u \\ -u\partial^{-1} \circ v & u\partial^{-1} \circ u \end{pmatrix} . \end{aligned}$$

Note that if $a_3 = b_3 = 0$, the above pair is such that H is "local" differential operator, and K is invertible. In this case, the Lenard-Magri recursion conditions give $\int h_{-1} = 0$ and $H \frac{\delta h_{n-1}}{\delta u} = K \frac{\delta h_n}{\delta u}$ for every $n \geq 0$. Namely, $\frac{\delta h_n}{\delta u} = 0$ for every n. Therefore, in this case, the corresponding Lenard-Magri scheme is of finite type, and we don't get any integrable system. Hence, we assume that $(a_3, b_3) \neq (0, 0)$.

Next, we need to find minimal fractional decompositions for H and K. This is given by the following

Lemma 5.1. *We have the following minimal fractional decomposition for the operator L_3:*

$$
\begin{pmatrix} v\partial^{-1} \circ v & -v\partial^{-1} \circ u \\ -u\partial^{-1} \circ v & u\partial^{-1} \circ u \end{pmatrix} = \begin{pmatrix} 0 & -uv \\ 0 & u^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{v}{u} & \frac{1}{u}\partial \circ u \end{pmatrix}^{-1}
$$

The rational matrix pseudodifferential operator H admits the fractional decomposition $H = AB^{-1}$ given by

$$
(5.2) \quad A = \begin{pmatrix} \partial - a_2\frac{v}{u} & -a_2\frac{1}{u}\partial \circ u - a_3 uv \\ \partial \circ \frac{v}{u} + a_2 & \partial \circ \frac{1}{u}\partial \circ u + a_3 u^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \frac{v}{u} & \frac{1}{u}\partial \circ u \end{pmatrix},
$$

which is minimal for $a_3 \neq 0$, while, for $a_3 = 0$, H is a matrix differential operator. The rational matrix pseudodifferential operator K admits the fractional decomposition $H = CB^{-1}$ given by with B as in (5.2) and

$$
(5.3) \qquad C = \begin{pmatrix} -\frac{v}{u} & -\frac{1}{u}\partial \circ u - b_3 uv \\ 1 & b_3 u^2 \end{pmatrix}.
$$

This decomposition is minimal for $b_3 \neq 0$, while, for $b_3 = 0$, $K = L_2$ is an invertible matrix.

Proof. Straightforward. □

As usual, in order to apply the Lenard-Magri scheme it is convenient to find the kernels of the operators B and C:

$$
\text{Ker } B = \mathcal{C} \begin{pmatrix} 0 \\ \frac{1}{u} \end{pmatrix}, \quad \text{Ker } C = \mathcal{C} \begin{pmatrix} -b_3 u \\ \frac{1}{u} \end{pmatrix}.
$$

We next compute the first few steps in the Lenard-Magri scheme. We have the following H and K-associations: $\int 0 \xleftrightarrow{H} P_0 \xleftrightarrow{K} \int h_0 \xleftrightarrow{H} P_1 \xleftrightarrow{K}$

$\int h_1 \xleftrightarrow{H} P_2$, where

(5.4)

$$P_0 = \alpha a_3 \begin{pmatrix} -v \\ u \end{pmatrix}, \quad \int h_0 = \frac{1}{2} \int (u^2 + v^2),$$

$$P_1 = \begin{pmatrix} u' - a_2 v \\ v' + a_2 u \end{pmatrix}, \quad \int h_1 = \int \left(uv' + \frac{a_2}{2}(u^2 + v^2) + \frac{b_3}{8}(u^2 + v^2)^2 \right),$$

$$P_2 = \begin{pmatrix} v'' + 2a_2 u' - a_2^2 v + \frac{b_3}{2}\left(u(u^2 + v^2)\right)' + \frac{a_3 - a_2 b_3}{2} v(u^2 + v^2) \\ -u'' + 2a_2 v' + a_2^2 u + \frac{b_3}{2}\left(v(u^2 + v^2)\right)' - \frac{a_3 - a_2 b_3}{2} u(u^2 + v^2) \end{pmatrix}.$$

Indeed, we have $P_0 = AF_0$, $BF_0 = 0$, for $F = \alpha \begin{pmatrix} 0 \\ \frac{1}{u} \end{pmatrix}$. We have $P_0 = CF_1$,

$\frac{\delta h_0}{\delta u} = BF_1$, for $F_1 = \begin{pmatrix} u \\ \frac{\beta}{u} \end{pmatrix}$, where $\alpha, \beta \in \mathcal{C}$ are chosen so that $\alpha a_3 - \beta b_3 = 1$

(we can always do so, since, by assumption, $(a_3, b_3) \neq (0,0)$). We have

$P_1 = AF_2$, $\frac{\delta h_0}{\delta u} = BF_2$, for $F_2 = \begin{pmatrix} u \\ 0 \end{pmatrix}$. We have $P_1 = CF_3$, $\frac{\delta h_1}{\delta u} = BF_3$,

for $F_3 = \begin{pmatrix} v' + a_2 u + \frac{b_3}{2} u(u^2 + v^2) \\ -\frac{1}{2}\frac{u^2 + v^2}{u} \end{pmatrix}$. And, finally, we have $P_2 = AF_4$,

$\frac{\delta h_1}{\delta u} = BF_4$, for $F_4 = F_3$.

Next, we check that the orthogonality conditions (2.17) hold for $N = 0$.

We have $F = \begin{pmatrix} f \\ g \end{pmatrix} \in \frac{\delta h_0}{\delta u}^\perp$ if and only if $\int (uf + vg) = 0$, namely if

$f = -\frac{v}{u} g + \frac{h'}{u}$, for some $h \in \mathcal{V}$. But in this case

$$F = \begin{pmatrix} -\frac{v}{u} g + \frac{h'}{u} \\ g \end{pmatrix} = C \begin{pmatrix} g + b_3 uh \\ -\frac{h}{u} \end{pmatrix} \in \operatorname{Im} C,$$

proving the first orthogonality condition (2.17). As for the first orthogonality condition, if $a_3 = 0$ there is nothing to prove since H is a matrix differential operator (i.e. the denominator is $\mathbb{1}$ in its minimal fractional decomposition).

If $a_3 \neq 0$, we can choose $\alpha = \frac{1}{a_3}$, and we have $F = \begin{pmatrix} f \\ g \end{pmatrix} \in P_0^\perp$ if and only

if $\int (-vf + ug) = 0$, namely if $g = \frac{v}{u} f + \frac{h'}{u}$, for some $h \in \mathcal{V}$. But in this case

$$F = \begin{pmatrix} f \\ \frac{v}{u} f + \frac{h'}{u} \end{pmatrix} = B \begin{pmatrix} f \\ \frac{h}{u} \end{pmatrix} \in \operatorname{Im} B,$$

proving the second orthogonality condition (2.17). Therefore, by Theorem 2.11, we deduce that the elements (5.4) can be extended to infinite sequences $\{\int h_n\}_{n \in \mathbb{Z}_+}$, $\{P_n\}_{n \in \mathbb{Z}_+}$, such that $\int h_{n-1} \xleftrightarrow{H} P_n \xleftrightarrow{K} \int h_n$.

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Finally, we have $|H| = 1$, $|K| = 0$, $\text{dord}(A) = 2$, $\text{dord}(B) = \text{dord}(C) = \text{dord}(D) = 1$, and $\text{dord}(P_2) = 2$. Hence, the inequality (2.23) holds. Therefore, by Proposition (2.15) we have $\text{dord}(P_n) = \text{dord}(\frac{\delta h_n}{\delta u}) = n$ for every $n \in \mathbb{Z}_+$. In particular, all the elements $\int h_n$'s and P_n's are linearly independent.

In conclusion, each equation of the hierarchy $\frac{du}{dt_n} = P_n$ is integrable, and the local functionals $\int h_n$'s are their integrals of motion. The first "non-trivial" equation of this hierarchy is for $n = 2$. Letting $a_2 = 0$, $a_3 = 2\alpha$, and $b_3 = 2\beta$, it has the form

$$(5.5) \qquad \begin{cases} \dfrac{du}{dt} = v'' + \alpha v(u^2 + v^2) + \beta\big(u(u^2 + v^2)\big)' \\ \dfrac{dv}{dt} = -u'' - \alpha u(u^2 + v^2) + \beta\big(v(u^2 + v^2)\big)' \end{cases}$$

If we view u and v as real valued functions, and we consider the complex valued function $\psi = u + iv$, the system (5.5) can be written as the following PDE:

$$i\frac{d\psi}{dt} = \psi'' + \alpha\psi|\psi|^2 + i\beta(\psi|\psi|^2)',$$

which, for $\beta = 0$, is the well-known Non-Linear Schroedinger equation (see e.g. [TF86, Dor93, BDSK09, DSK12]).

It is not difficult to show that when going back the Lenard-Magri scheme is "blocked" at $\int h_{-2}$ when $a_3 = 0$, and it is of finite type when $a_3 \neq 0$. Hence, we don't get any non-evolutionary PDE in this case.

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